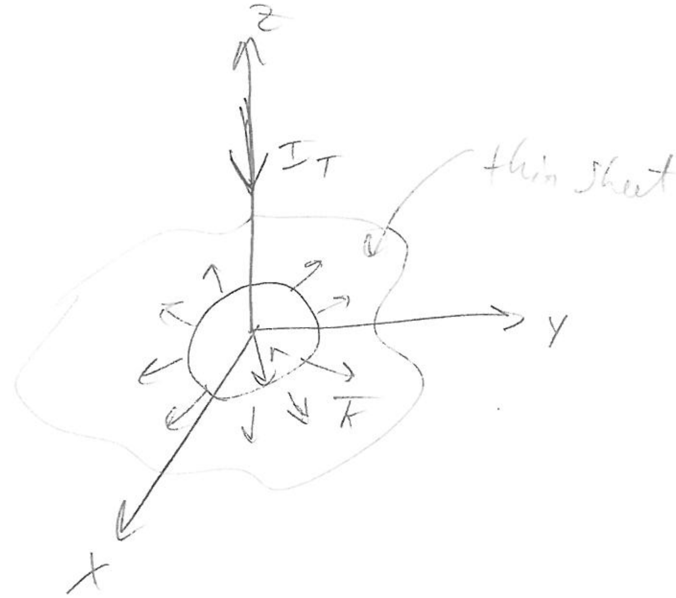
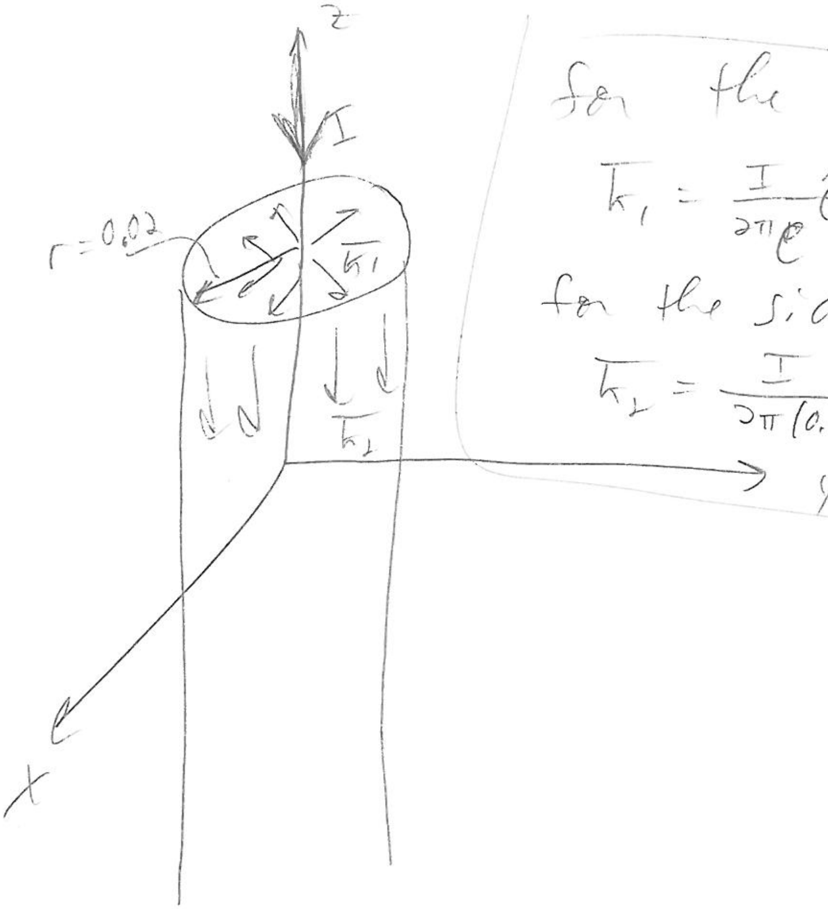
6.17) A current I_T follows a filament down the z axis and enters a thin conducting sheet at $z=0$. Express $\bar{K}$ for this sheet?



$$\bar{K} = \frac{I_T}{2\pi r}\,\hat{\rho}$$

because when the current I_T hits the sheet, it spreads evenly across it in the $\hat{\rho}$ direction

6.19) A current I enters a thin right circular cylinder at the top. Express $\bar{K}$ if the radius of the cylinder is 2 cm.



for the top

$$\bar{K}_1 = \frac{I}{2\pi\rho}\hat{\rho}$$

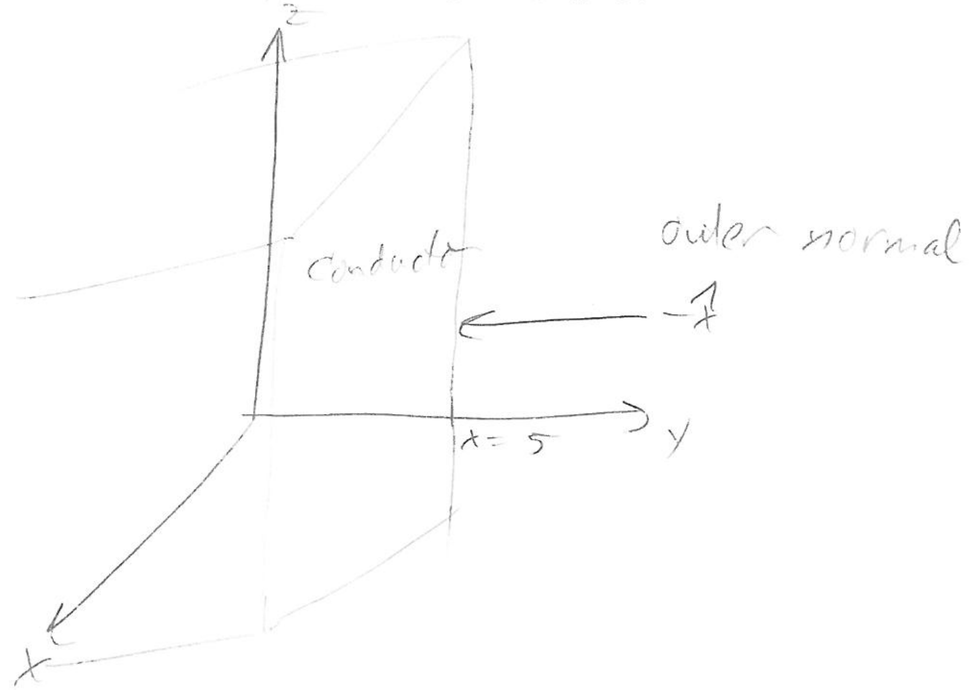
for the side

$$\bar{K}_2 = \frac{I}{2\pi(0.02)}(-\hat{z})$$

6.21) A conductor occupying the region $x \geq 5$ has a surface charge density

$$\rho_s = \frac{\rho_0}{\sqrt{y^2 + z^2}}$$

Write the expressions for E and $\bar{D}$ just outside the conductor



- for an infinite plane

$$\bar{E} = \frac{\rho_s}{2\varepsilon_0}\hat{n}$$

$$\bar{D} = D_n(-\hat{x}) = \rho_s(-\hat{x}) = \frac{\rho_0}{\sqrt{y^2+x^2}}(-\hat{x})$$

$$E = \frac{\bar{D}}{\varepsilon_0} = \frac{\rho_0}{\varepsilon_0\sqrt{y^2+x^2}}(-\hat{x})$$

Ch 7: Capacitance and Dielectric Materials

7.1) Find the polarization $\overline{P}$ in a dielectric material with $\varepsilon_r = 2.8$ if $\overline{D} = (3E-7)\hat{a}$ C/m^2

$$\overline{P} = \chi_e \varepsilon_0 \overline{E}$$

$$\chi_e = \varepsilon_r - 1 = 2.8 - 1 = 1.8 = \chi_e$$

$$\overline{D} = \varepsilon_0 \varepsilon_r E \Rightarrow E = \frac{D}{\varepsilon_0 \varepsilon_r} = \frac{(3E-7)}{2.8(8.854E-12)}\vec{a}$$

$$E = 12.1E3\,\vec{a}$$

$$\Rightarrow \overline{P} = (1.8)(12.1E3)(8.854E-12)\vec{a}$$

$$\boxed{\overline{P} = 1.93E-7\,\hat{a}}$$

7.3) 2 pt charges in a dielectric medium where $\varepsilon_r = 5.2$ interact w/ a force $F = 8.6E-3$ N, what force could be expected if it were free space?

$$F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 \varepsilon_r |\overline{R}_{21}|}\overline{R_{21}}$$

$$\varepsilon_r ; \quad 8.63E-3 = \frac{F}{\varepsilon_0 \cancel{\varepsilon_r}}_{\;1} \quad \xrightarrow{\varepsilon_r = 1} \quad F = 5.3(8.63E-3)$$

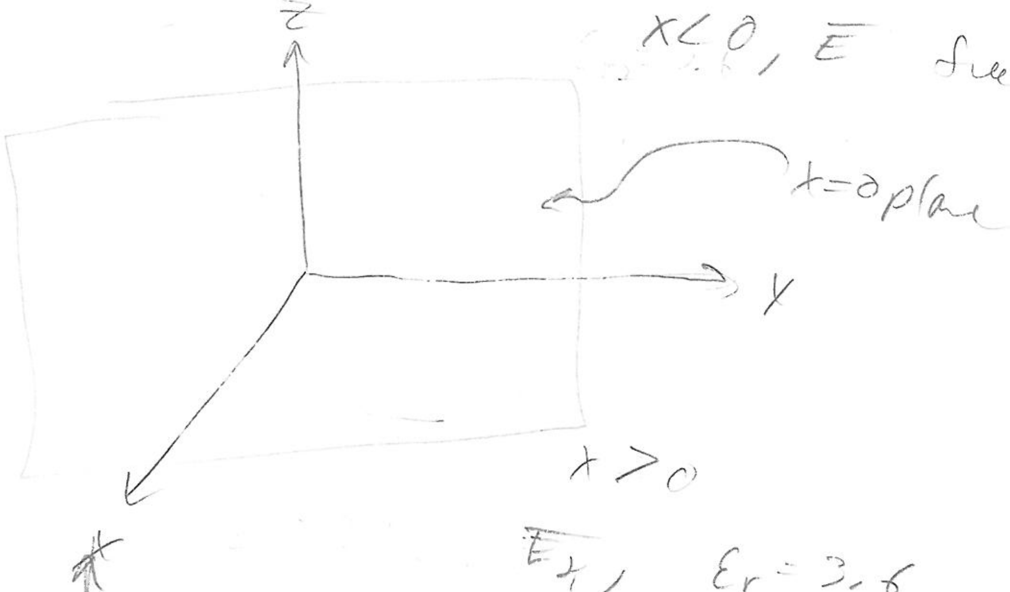
$$\boxed{F = 45.74E-3\ N}$$

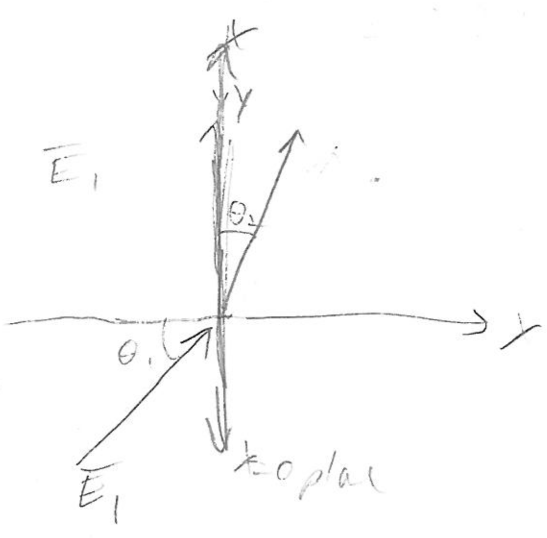
7.5) In the free-space region $x < 0$ the electric field intensity is

$\bar{E}_1 = 3\hat{x} + 5\hat{y} - 3\hat{z}$ (V/m).

The region $x > 0$ is a dielectric for which $\varepsilon_{r2} = 3.6$.

Find the angle θ_2 that the field in the dielectric makes with the $x=0$ plane.





1) Tangential comp. of $\overline{E}$ is cont accross the interface

$E_{t_2} = E_{t_1} = 5\hat{y} - 3\hat{z}$ because interface is $x=0$ plane

$\overline{E}_1 = 3\hat{x} + 5\hat{y} - 3\hat{z}$

$\overline{E}_2 = E_{x2}\hat{x} + 5\hat{y} - 3\hat{z}$ (eq 2)

2) The normal comp. of $\overline{D}$ has a discontinuity accross the interface

$D_1 = \varepsilon_0 \varepsilon_{r1} E_1 = (1)\varepsilon_0 \left[3\hat{x} + 5\hat{y} - 3\hat{z}\right]$

$\overline{D}_1 = 3\varepsilon_0\hat{x} + 5\varepsilon_0\hat{y} - 3\varepsilon_0\hat{z}$

$D_2 = \varepsilon_0 \varepsilon_{r2} E_2 = (3.6)\varepsilon_0 \left[E_x\hat{x} + E_y\hat{y} + E_z\hat{z}\right]$

$\overline{D}_2 = D_{x2}\hat{x} + D_{y2}\hat{y} + D_{z2}\hat{z}$

- Since the interface has no free charges, the normal comp's to the interface for $\overline{D}$ are equal

$D_{n1} = D_{n2}$, thus

$D_{x2} = D_{x1} = 3\varepsilon_0$

so $\overline{D}_2 = 3\varepsilon_0\hat{x} + D_{y2}\hat{y} + D_{z2}\hat{z}$ (eq 1)

→ So, apply the relation $\overline{D_2} = \varepsilon_0 \varepsilon_{r2} \overline{E_2}$ to eq's 1 and 2 to solve for the unknown components:

$$\overline{D_2} = \varepsilon_0 \varepsilon_{r2} \overline{E_2}$$

$$3\varepsilon_0 \hat{x} + D_{y2}\hat{y} + D_{z2}\hat{z} = 3.6\varepsilon_0 \left[E_{x2}\hat{x} + 5\hat{y} - 3\hat{z}\right]$$

$$3\varepsilon_0 \hat{x} + D_{y2}\hat{y} + D_{z2}\hat{z} = 3.6\varepsilon_0 E_{x2}\hat{x} + 3.6\varepsilon_0(5)\hat{y} - 3.6\varepsilon_0(3)\hat{z}$$

$$\Rightarrow 3\varepsilon_0 \hat{x} = 3.6\varepsilon_0 E_{x2}\hat{x}$$

$$\Rightarrow E_{x2} = \frac{3\varepsilon_0}{3.6\varepsilon_0} = \frac{3}{3.6} = E_{x2}$$

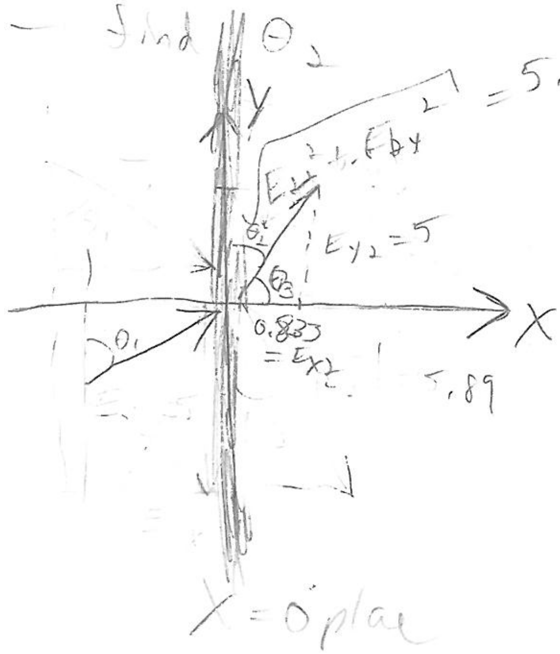
$$\Rightarrow D_{y2}\hat{y} = 3.6\varepsilon_0(5)\hat{y}$$

$$\Rightarrow D_{y2} = 18\varepsilon_0$$

$$\Rightarrow D_{z2}\hat{z} = -3.6\varepsilon_0(3)\hat{z}$$

$$\Rightarrow D_{z2} = -10.8\varepsilon_0$$

– find θ_2



$= 5.07$ $\quad E_{2x} = \frac{3}{3.6} \approx 0.833, \quad E_{2y} = 5$

$$|\overline{E_2}| = \sqrt{\left(\frac{3}{3.6}\right)^2 + 5^2 + (-3)^2} = 5.89$$

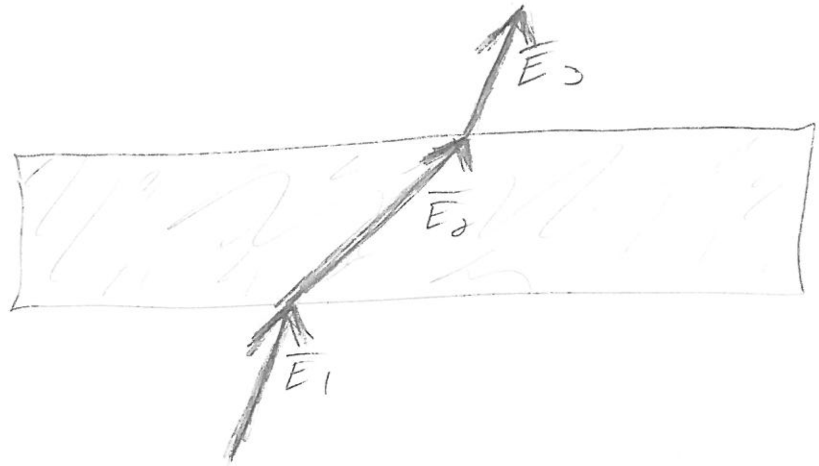
$\sin = \frac{opp}{hyp} \Rightarrow \sin\theta_3 = \frac{5}{5.07}$

$\tan = \frac{opp}{adj}$

$\Rightarrow \theta_3 = 80.5°$

$\Rightarrow \theta_2 = 90 - \theta_3 = 90 - 80.5 = 9.5°$

7.7) A planar dielectric slab with free space on either side. Assuming a constant field $\overline{E}_2$ within the slab, show that $\overline{E}_3 = \overline{E}_1$



— By continuity of E_t across the two interfaces:

$E_{t3} = E_{t1}$

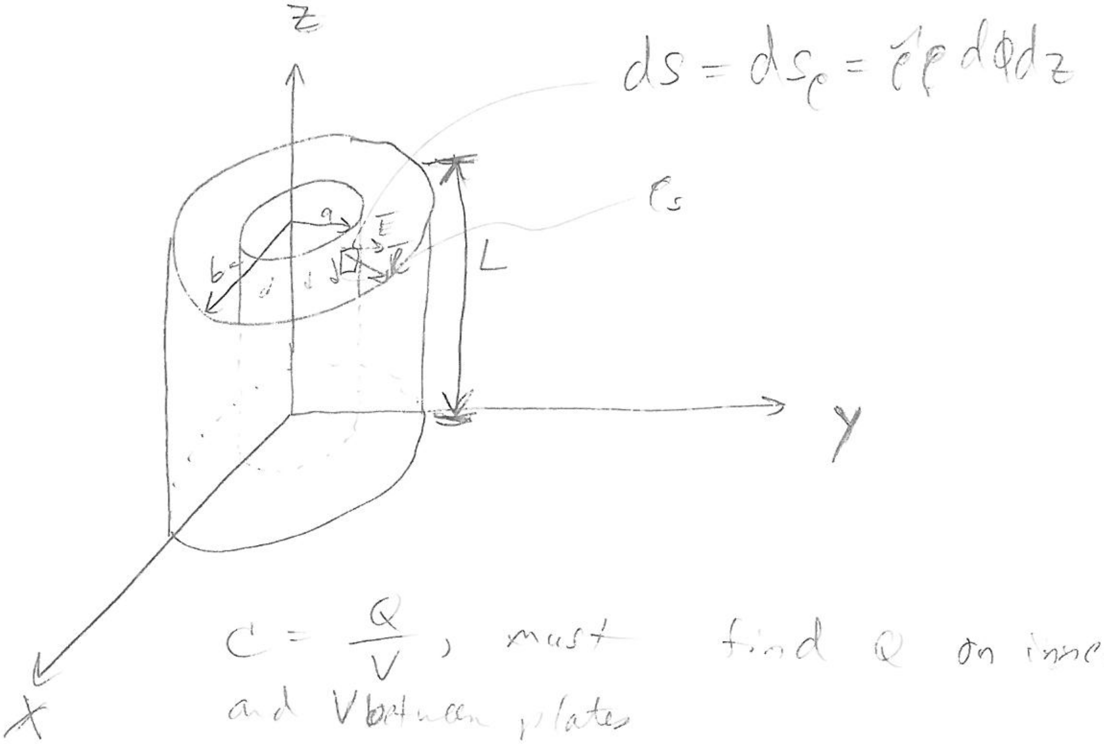
— By continuity of D_n across the two interfaces:

$D_{n3} = D_{n1}$ and $E_{n3} = E_{n1}$

— Consequently

$\overline{E}_3 = \overline{E}_1$

7.9) Find the capacitance of a coaxial cap of length L, where the inner conductor has radius a and the outer radius b



$C = \frac{Q}{V}$, must find Q on inner plate, and V between plates

- assume a surface charge density of ρ_s on the inner cylinder r=a, find the E and $\overline{D}$ feilds due to the inner cylinder
  - by symmetry, the feild between the cylinders must be radial and a function of ρ only. Thus, for $a < \rho < b$

$$D_r \propto \frac{1}{\rho} \Rightarrow D_r = \frac{\rho_s}{\rho} \Rightarrow \overline{D} = \frac{\rho_s}{\rho}\hat{\rho}$$

$$\cancel{\Rightarrow \nabla \cdot \overline{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\rho_s}{\rho}\right) = 0}$$

$$\Rightarrow E = \frac{\overline{D}}{\varepsilon} = \frac{\rho_s}{\varepsilon\rho}\hat{\rho} = E$$

- next, we use $\bar{E}$ to find V_{ab}, the voltage between the cylinders

$$V_{ab} = -\int_b^a E \cdot dl = -\int_b^a \frac{\rho_s}{\varepsilon\rho}\hat{\rho}\cdot\hat{\rho}\, d\rho$$

$$= \frac{-\rho_s}{\varepsilon}\int_b^a \frac{1}{\rho}\, d\rho = \frac{-\rho_s}{\varepsilon}\Big[\ln \rho\Big]_b^a = \frac{-\rho_s}{\varepsilon}\Big[\ln a - \ln b\Big]$$

$$V_{ab} = \frac{\rho_s}{\varepsilon}\ln\left(\frac{b}{a}\right)$$

- finally, find the charge on the inner cylinder

$$\rho_s = \frac{Q_s}{A} \Rightarrow Q_s = \rho_s A = \rho_s 2\pi a L = Q_s$$

$$A = 2\pi a L$$

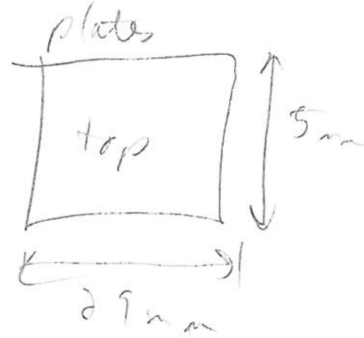
- apply the cap formula $C = \frac{Q}{V}$

$$C = \frac{Q}{V} = \frac{\cancel{\rho_s} 2\pi a L}{\frac{\cancel{\rho_s}}{\varepsilon}\ln\left(\frac{b}{a}\right)} = \frac{2\pi a L \varepsilon}{\ln\left(\frac{b}{a}\right)} = C$$

7.11) Find the separation d which results in the same capacitance, $7.76pF$, when the plates are $\parallel$ w/ the same dielectric

$\varepsilon_r = 4.5$

$C = 7.76pF$



- use the formula

$$C = \frac{\varepsilon_0 \varepsilon_r A}{d}$$

$\Rightarrow A = (5E-3)(29E-3)$

$A = 145E-6 \, m^2$

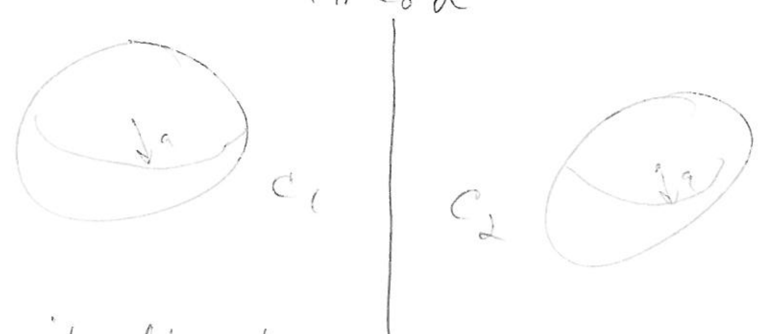
$$7.76E-12 = \frac{4.5(8.854E-12)(145E-6)}{d}$$

$$\Rightarrow \boxed{d = 0.744 \text{ mm}}$$

7.13) Find the cap. between 2 spherical shells of radius a separated by a distance $d \gg a$

- use the cap. of a spherical shell formula:

$$C = 4\pi \varepsilon_0 a$$



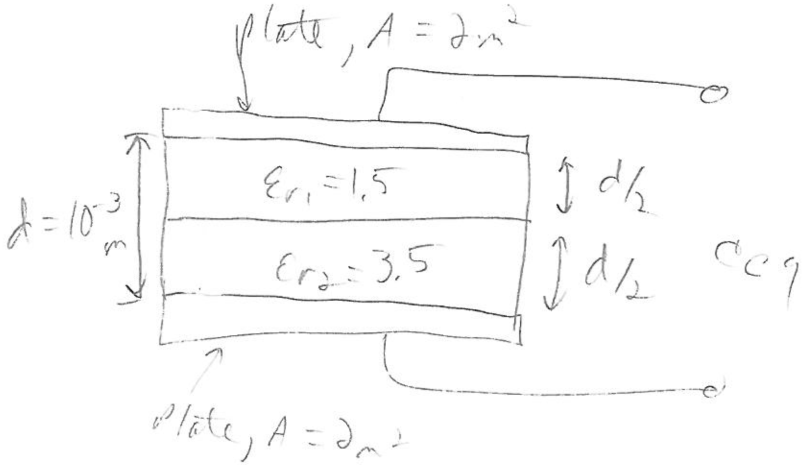
its like having 2 caps in series, so

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$C_1 = 4\pi\varepsilon_0 a, \quad C_2 = 4\pi\varepsilon_0 a$$

$$\Rightarrow \frac{1}{C_{eq}} = \frac{2}{4\pi\varepsilon_0 a} \Longrightarrow C_{eq} = \frac{4\pi\varepsilon_0 a}{2}$$

7.15) Find the cap of a multi dielectric cap



– this is like having 2 caps in series

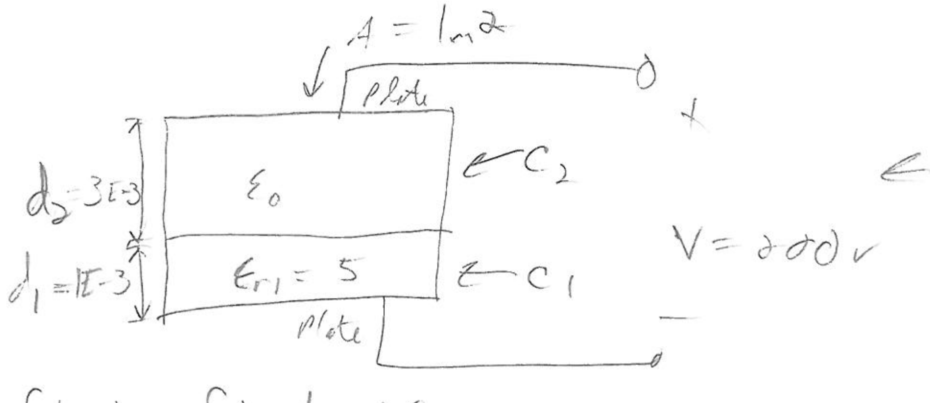
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$$

$$C_1 = \frac{\varepsilon_0 \varepsilon_{r1} A}{d/2} = \frac{(8.854E-12)(1.5)2}{10^{-3}/2} = \cancel{26.6\,nF} \quad 53.1\,nF$$

$$C_2 = \frac{\varepsilon_0 \varepsilon_{r2} A}{d/2} = \frac{(8.854E-12)(3.5)2}{10^{-3}/2} = \cancel{62.0\,nF} \quad 123.96\,nF$$

$$C_{eq} = \frac{(\overset{53.1n}{\cancel{26.6\,nF}})(\overset{123.96n}{\cancel{62.0\,nF}})}{\underset{53.1n}{\cancel{26.6\,nF}} + \underset{123.96n}{\cancel{62.0n}}} = \cancel{18.6\,nF} \quad 37.17\,nF$$

7.17) Find the voltage accross each dielectric in the cap when the applied $V = 200v$



- first, find the capacitance values accross each cap, then determine C_{eq}

$$C_1 = \frac{\varepsilon_0 \varepsilon_{r1} A}{d_1} = \frac{(8.854E\text{-}12)(5)\,1}{1E\text{-}3} = 44.27\,nF$$

$$C_2 = \frac{\varepsilon_0 A}{d_2} = \frac{(8.854E\text{-}12)\,1}{3E\text{-}3} = 2.95\,nF$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$$

$$C_{eq} = \frac{(44.27n)(2.95n)}{44.27n + 2.95n} = 2.77\,nF = C_{eq}$$

- Next apply the cap formula, and find the charge Q

$$C = \frac{Q}{V} \Rightarrow Q = CV = (2.77n)\,200 = 0.554\,\mu C = Q$$

→ 10

- looking at C_1

$C_1 = 44.27\,nF$

$C_1 = \frac{Q}{V_1} \Rightarrow 44.27n = \frac{0.554\mu C}{V}$

$\Rightarrow V_1 = 12.5\,V$

- looking at C_2

$C_2 = 2.77\,nF$

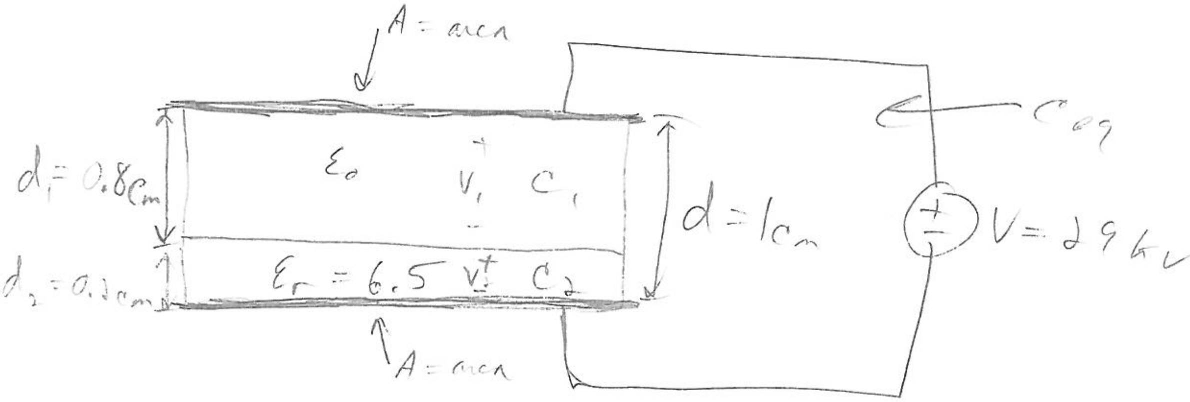
done

$\Rightarrow 2.77nF = \frac{0.554\mu C}{V_2} \Rightarrow V_2 = 187.797\,V$

7.19) A free space ‖ plate cap is charged by a momentary connection to a voltage source V, which is then removed. Determine how W_E, D, E, C, and V change as the plates are moved apart to a separation distance $d_2 = 2d_1$:

Relationship	Explanation
$D_2 = D_1$	$D = Q/A$
$E_2 = E_1$	$E = D/\varepsilon_0$
$W_{E_2} = W_{E_1}$	$W_E = \frac{1}{2}\int \varepsilon_0 E^2\,dv$, and the vol is doubled
$C_2 = \frac{1}{2}C_1$	$C = \varepsilon_0 A/d$
$V_2 = 2V_1$	$V = Q/C$

7.21) A ∥ plate cap w/a separation of $d = 1.0$ cm has 29 kV applied when free space is the only dielectric. Assume that air has a dielectric strength $E_{max} = 30$ kV/cm. Show why the air breaks down when a thin piece of glass ($\varepsilon_r = 6.5$) w/a dielectric strength $E_{max} = 290$ kV/cm and thickness $d_2 = 0.2$ cm is inserted:



- first find the capacitance of each cap

$$C_1 = \frac{\varepsilon_0 A}{d_1} = \frac{\varepsilon_0 A}{8E-3} = 125\,\varepsilon_0 A = C_1$$

$$C_2 = \frac{\varepsilon_0 \varepsilon_r A}{d_2} = \frac{\varepsilon_0 (6.5) A}{2E-3} = 3250\,\varepsilon_0 A = C_2$$

- find the equivalent cap

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = \frac{(125\,\varepsilon_0 A)(3250\,\varepsilon_0 A)}{125\,\cancel{\varepsilon_0 A} + 3250\,\cancel{\varepsilon_0 A}} = \boxed{120.4\,\varepsilon_0 A = C_{eq}}$$

2

- find the charge Q

$$C_{eq} = \frac{Q}{V} \Rightarrow 120.4\varepsilon_0 A = \frac{Q}{29kV}$$

$$\Rightarrow Q = (3.49E6)\varepsilon_0 A$$

- find the voltage across C_1

$$C_1 = \frac{Q}{V_1} \Rightarrow 125\cancel{\varepsilon_0 A} = \frac{(3.49E6)\cancel{\varepsilon_0 A}}{V_1}$$

$$\Rightarrow V_1 = 27.92kV$$

- finally, find E_1, the E feild value inside gap 1 and see if it exceeds $E_{max} = 30\,kV/cm$

$$E_1 = \frac{V}{m} = \frac{27.92k}{0.8cm} = 34.9kV/cm > 30kV/cm$$

$m = d_1 = 0.008$

that is why the air will break down

Ch 8: Laplace's Equation

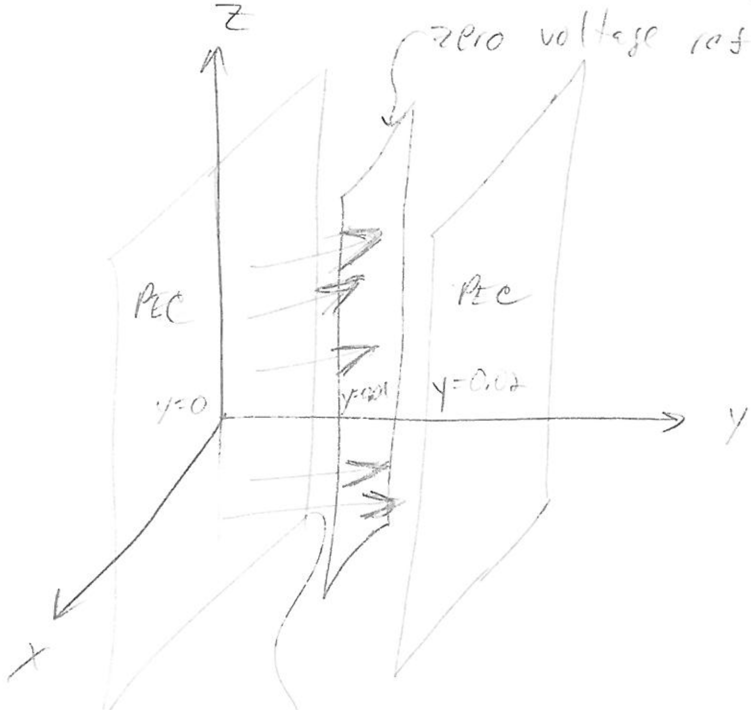
8.1) The potential has the value V_1 on $\frac{1}{n}$ of the circle, and the value 0 on the rest of the circle. Find the potential at the center of the circle. (The entire region is charge-free)

$$V_0 = \frac{V_1}{n}$$

8.3) Prove that within a charge-free region the potential cannot attain a maximum value;

- Suppose that a maximum were attained at an interior point P. Then a very small sphere could be centered on P, such that the potential V_0 at P exceeded the potential at each point on the sphere. Then V_0 would also exceed the average value of the potential over the sphere. But that would contradict the mean value theorem.

8.5) 2 parallel conducting planes in free space are at $y=0$ and $y=0.02$, and the zero voltage reference is at $y=0.01$. If $\bar{D} = (253E-9)\hat{y}\ \frac{C}{m^2}$ between the conductors, determine the conductor voltages.



$\bar{D} = (253E-9)\hat{y}\ \frac{C}{m^2}$

$$\frac{\partial^2 V}{\partial y^2} = 0 \quad \text{integrate both sides}$$

$$\Rightarrow V = Ay + B \qquad (1)$$

- from maxwell's eqs $E = -\nabla V$

$$E = -\nabla\left[Ay + B\right] = -A\hat{y} = E$$

but also $\bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{253E-9}{8.854E-12}\hat{y} = 28.6E3\,\hat{y} = E$

$$E = -A\hat{y} = 28.6E3\,\hat{y} = E$$

$$\Rightarrow A = -28.6E3$$

- looking at eq 1, apply the A result from above

$$V = Ay + B, \quad A = -28.6E3$$

$$V = -28.6E3\,y + B$$

- Solve for B by setting V=0, V is only zero at position y = 0.01, thus

$$0 = -28.6E3(0.01) + B$$

$$\Rightarrow B = 286$$

- So finally, we can find the potential everywhere

$$\boxed{V = -28.6E3\,y + 286}$$ done

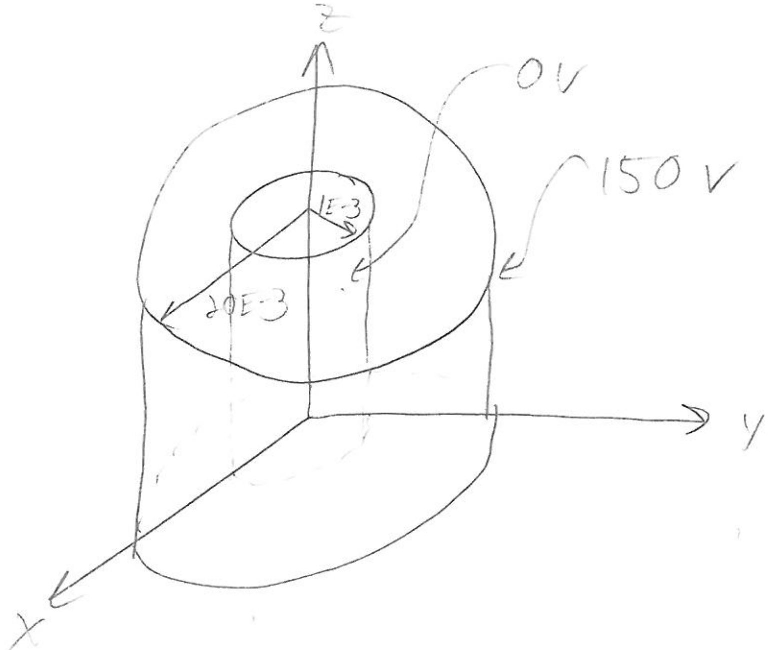
- So at y = 0

$$V = -28.6E3(0) + 286 = \boxed{286\,V}$$

- and, at y = 0.02

$$V = -28.6E3(0.02) + 286 = \boxed{-286\,V}$$

8.7) Find the potential function and the electric field intensity for the region between 2 concentric right circular cylinders, where $V = 0$ at $r = 1E-3$, and $V = 150$ V at $r = 20E-3$. (neglecting fringing)



- Since it is a cylinder, V will be in the radial direction ρ

so $0 = \nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right)$ in cylindrical coords

- apply the laplacian

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

- integrate both sides once

$$\rho \cdot \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0 \cdot \rho$$

$$\int \frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) d\rho = \int 0 \, d\rho$$

$$\rho \frac{\partial V}{\partial \rho} = A$$

- integrate again

$$\frac{\rho \frac{\partial V}{\partial \rho}}{\rho} = A \cdot \frac{1}{\rho}$$

$$\int \frac{\partial V}{\partial \rho} d\rho = \int \frac{A}{\rho} d\rho$$

$$V = A \int \frac{1}{\rho} d\rho$$

$$\boxed{V = A \ln \rho + B} \quad (1)$$ is the voltage eq

- ~~from Maxwell's eq~~

$$\bar{E} = -\nabla V$$

$$\bar{E} - \nabla V = -\left[\hat{\rho} \frac{\partial V}{\partial \rho} + \cancel{\hat{\phi} \frac{1}{\rho} \frac{\partial f}{\partial \phi}} + \cancel{\hat{z} \frac{\partial f}{\partial z}}\right]$$

$$= -\hat{\rho} \frac{\partial}{\partial \rho}\left[A \ln \rho + B\right] = \boxed{-\frac{A}{\rho} \hat{\rho} = \bar{E}} \quad (2)$$

- solve for A and B in eq (1)
  - looking at $\rho = 1E-3\,m$, $V = 0$

$$0 = A \ln(1E-3) + B \quad (3)$$

  - looking at $\rho = 20E-3$, $V = 150$

$$150 = A \ln(20E-3) + B \quad (4)$$

- Solve for A and B

$$0 = A \ln(1E-3) + B$$
$$150 = A \ln(20E-3) + B$$
$$\Rightarrow B = 150 - A \ln(20E-3)$$

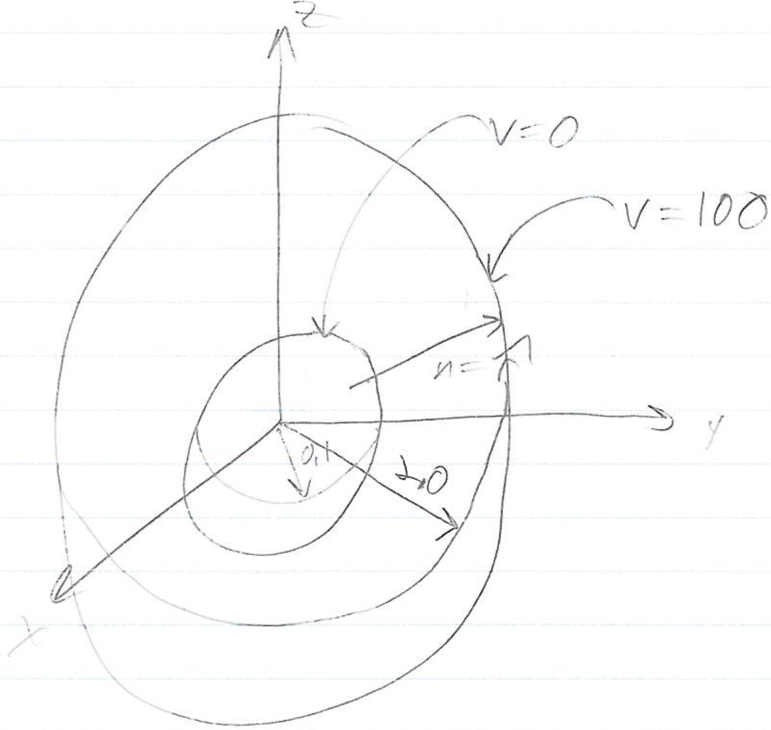
$$\Rightarrow 0 = A \ln(1E-3) + 150 - A \ln(20E-3)$$

$$0 = A\left[\ln(1E-3) - \ln(20E-3)\right] + 150$$
$$-150 = A \ln\left[\frac{1E-3}{20E-3}\right] \Rightarrow A = 50.1$$
$$B = 150 - 50.1 \ln(20E-3) = 345.99 = B$$

- so $\boxed{V = 50.1 \ln c + 346}$ done

- and $\boxed{E = -\frac{50.1}{c} \hat{c}}$ done

8.9) In spherical coordinates, $V = 0$ for $r = 0.1$ and $V = 100$ for $r = 2.0$. Assuming free space between these concentric spherical shells, find $\bar{E}$ and $\bar{D}$:



- Since the normal vector is $\hat{r}$, this is a 1D spherical problem
- apply the laplacian

$$\nabla^2 V = 0$$

$$r^2 \cdot \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0 \cdot r^2$$

$$\int \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) dr = \int 0 \, dr$$

$$\frac{1}{r^2} \cdot \quad r^2 \frac{\partial V}{\partial r} = A \cdot \frac{1}{r^2}$$

$$\int \frac{\partial v}{\partial r} dr = \int \frac{A}{r^2} dr$$

$$V = -\frac{A}{r} + B \quad (1)$$

- for $r = 0.1$, $V = 0$

$$0 = -\frac{A}{0.1} + B$$

$$0 = -10A + B \quad (2)$$

- for $r = 2$, $V = 100$

$$100 = -\frac{A}{2} + B \quad (3)$$

- use eq's 2 and 3 and solve for A and B, then get the final solution to V

$$0 = -10A + B$$
$$100 = -\frac{A}{2} + B$$

$$\Rightarrow B = 100 + \frac{A}{2}$$

$$0 = -10A + 100 + \frac{A}{2}$$

$$0 = A\left(-10 + \frac{1}{2}\right) + 100$$

$$A(9.5) = 100 \Rightarrow A = 10.52$$

$$B = 100 + \frac{10.52}{2} = 105.26 = B$$

$$\boxed{V = -\frac{10.52}{r} + 105.26}$$

- find E

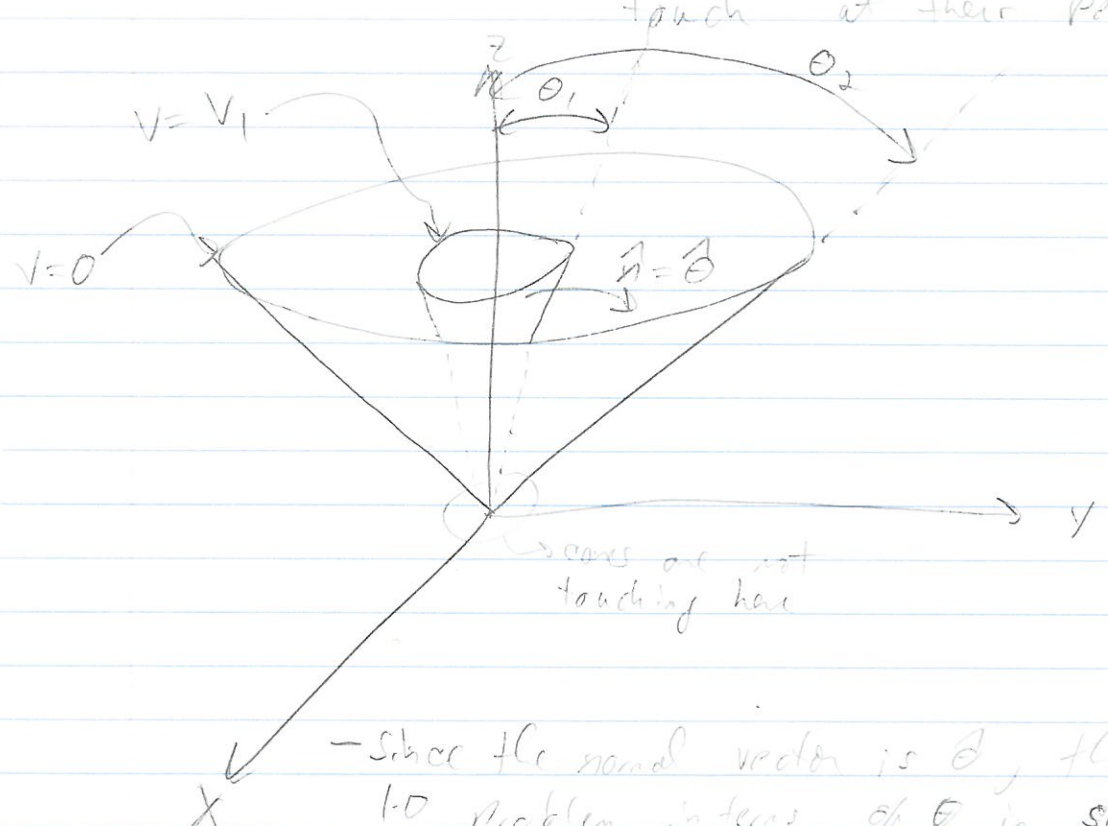
$$\bar{E} = -\nabla V = -\hat{r}\frac{\partial v}{\partial r} = -\hat{r}\frac{\partial}{\partial r}\left[-\frac{9.52}{r} + 95.24\right]$$

$$\boxed{\bar{E} = -\hat{r}\left[\frac{+10.52}{r}\right]}$$

$$\bar{D} = \varepsilon_0 \bar{E} = \boxed{-\hat{r}\,\varepsilon_0\left[\frac{10.52}{r}\right] = \bar{D}}$$

8.11] Solve laplace's equation for the region between coaxial cones. A potential V_1 is assumed at θ_1, and $V=0$ at θ_2. The cone vertices are insulated at $r=0$.

→ this means the cones do not touch at their peaks



- Since the normal vector is θ, this is a 1-D problem in terms of θ in spherical coords
- apply laplace's eq

$$\nabla^2 V = 0$$

$r^2 \sin\theta \cdot$

$$\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0 \quad \cdot r^2 \sin\theta$$

$$\int \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) d\theta = \int 0 \, d\theta$$

$$\frac{1}{\sin\theta} \cdot \sin\theta \frac{dV}{d\theta} = A \cdot \frac{1}{\sin\theta}$$

$$\int \frac{dV}{d\theta} d\theta = \int \frac{A}{\sin\theta} d\theta$$

$$V = A \int \frac{1}{\sin\theta} d\theta = A \ln\left[\tan\frac{\theta}{2}\right] + B$$

$$V = A \ln\left[\tan\left(\frac{\theta}{2}\right)\right] + B \qquad (1)$$

- looking at pos θ_1, $V = V_1$

$$V_1 = A \ln\left[\tan\left(\frac{\theta_1}{2}\right)\right] + B$$

- looking at pos θ_2, $V = 0$

$$0 = A \ln\left[\tan\left(\frac{\theta_2}{2}\right)\right] + B$$

$$\Rightarrow B = -A \ln\left[\tan\frac{\theta_2}{2}\right] \qquad (2)$$

- Solve for A

$$V_1 = A \ln\left[\tan\left(\frac{\theta_1}{2}\right)\right] - A \ln\left[\tan\frac{\theta_2}{2}\right]$$

$$V_1 = A\left[\ln\left(\tan\frac{\theta_1}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)\right]$$

$$V_1 = A \ln\left[\frac{\tan\left(\frac{\theta_1}{2}\right)}{\tan\frac{\theta_2}{2}}\right] \Rightarrow A = \frac{V_1}{\ln\left[\frac{\tan\left(\frac{\theta_1}{2}\right)}{\tan\frac{\theta_2}{2}}\right]} \qquad (3)$$

- plug eq 3 into 2, solve for B

$$B = -A \ln\left[\tan \frac{\theta_2}{2}\right]$$

$$B = \frac{-V_1 \ln\left[\tan \frac{\theta_2}{2}\right]}{\ln\left[\frac{\tan \frac{\theta_1}{2}}{\tan \frac{\theta_2}{2}}\right]} \quad (4)$$

- put (3) and (4) back into (1)

$$V = A \ln\left[\tan \frac{\theta}{2}\right] + B$$

$$V = \left[\frac{V_1}{\ln\left[\frac{\tan \frac{\theta_1}{2}}{\tan \frac{\theta_2}{2}}\right]}\right] \ln\left[\tan \frac{\theta}{2}\right] - \frac{V_1 \ln\left[\tan \frac{\theta_2}{2}\right]}{\ln\left[\frac{\tan \frac{\theta_1}{2}}{\tan \frac{\theta_2}{2}}\right]}$$

$$V = \frac{V_1 \ln\left(\tan \frac{\theta}{2}\right)}{\ln\left(\tan \frac{\theta_1}{2}\right) - \ln\left(\tan \frac{\theta_2}{2}\right)} - \frac{V_1 \ln\left(\tan \frac{\theta_2}{2}\right)}{\ln\left(\tan \frac{\theta_1}{2}\right) - \ln\left(\tan \frac{\theta_2}{2}\right)}$$

$$\boxed{V = \frac{V_1\left[\ln\left(\tan \frac{\theta}{2}\right) - \ln\left(\tan \frac{\theta_2}{2}\right)\right]}{\ln\left(\tan \frac{\theta_1}{2}\right) - \ln\left(\tan \frac{\theta_2}{2}\right)}}$$

8.13) Using the results from the previous problem, find the charge distribution on the conducting plane at $\theta_2 = 90°$, given $\theta_1 = 10°$, $V_1 = 100\,V$

$$V = \frac{100}{\ln\left(\frac{\tan 5°}{\tan 45°}\right)} \ln\left(\tan\frac{\theta}{2}\right) - \cancel{\frac{100\ln\left(\tan 45°\right)}{\ln\left(\frac{\tan 5°}{\tan 45°}\right)}}^{0}$$

$$V = -41\ln\left(\tan\frac{\theta}{2}\right)$$

- find $\bar{E}$

$$\bar{E} = -\nabla V = -\hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta}$$

$$= -\hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\left[-41\ln\left(\tan\frac{\theta}{2}\right)\right]$$

$$= -\hat{\theta}\frac{1}{r}\left[\frac{-41}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$\bar{E} = \hat{\theta}\frac{41}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\Rightarrow \bar{D} = \varepsilon_0\bar{E} = \frac{\hat{\theta}\,41\varepsilon_0}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \bar{D}$$

$\rightarrow$

$$V = \frac{-\rho_v}{2\varepsilon}\int \rho \, d\rho + A\int \frac{1}{\rho}\, d\rho$$

$$\boxed{V = \frac{-\rho_v \rho^2}{4\varepsilon} + A\ln\rho + B} \quad \text{disc}$$

8.17) A potential in cylindrical coordinates is a function of ρ and ϕ but not z. Obtain the separated diff. eq's for R and Φ, where $V = R(\rho)\Phi(\phi)$, and solve them. (The region is charge-free)

$\nabla^2 V = 0$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

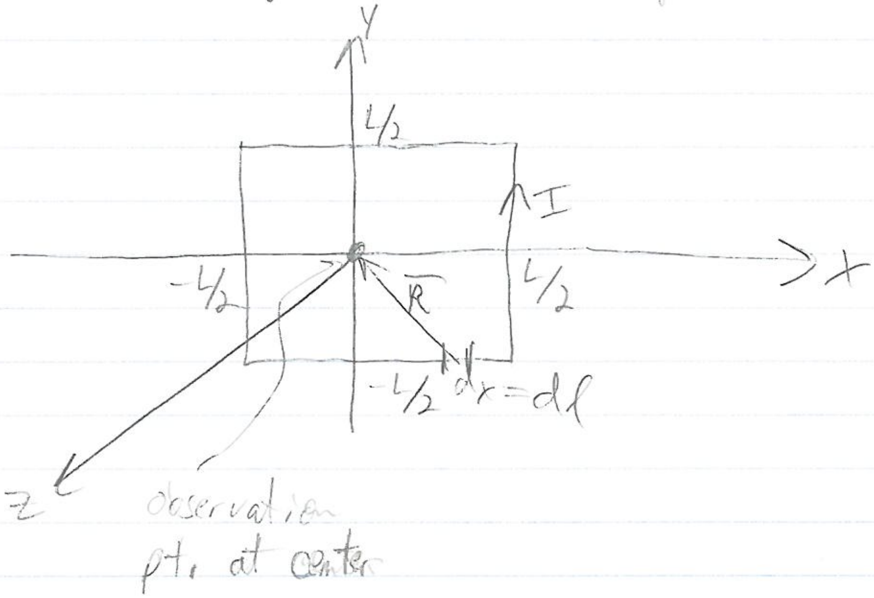
$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\left[R(\rho)\Phi(\phi)\right]\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\left[R(\rho)\Phi(\phi)\right] = 0$$

$$\Rightarrow \Phi \frac{\partial^2 R}{\partial \rho^2} + \frac{\Phi}{\rho}\frac{\partial R}{\partial \rho} + \frac{R}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{r}{R}\frac{\partial R}{\partial \rho} = \frac{-1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}$$

Ch 9: Ampere's Law and the Magnetic Field

9.1) Find $\overline{H}$ at the center of a square current loop of side length L



– from symmetry, there are a total of 4 sides, giving us 8 half sides. If we take the line integral over just one side, we can multiply that by 8 to get the entire result

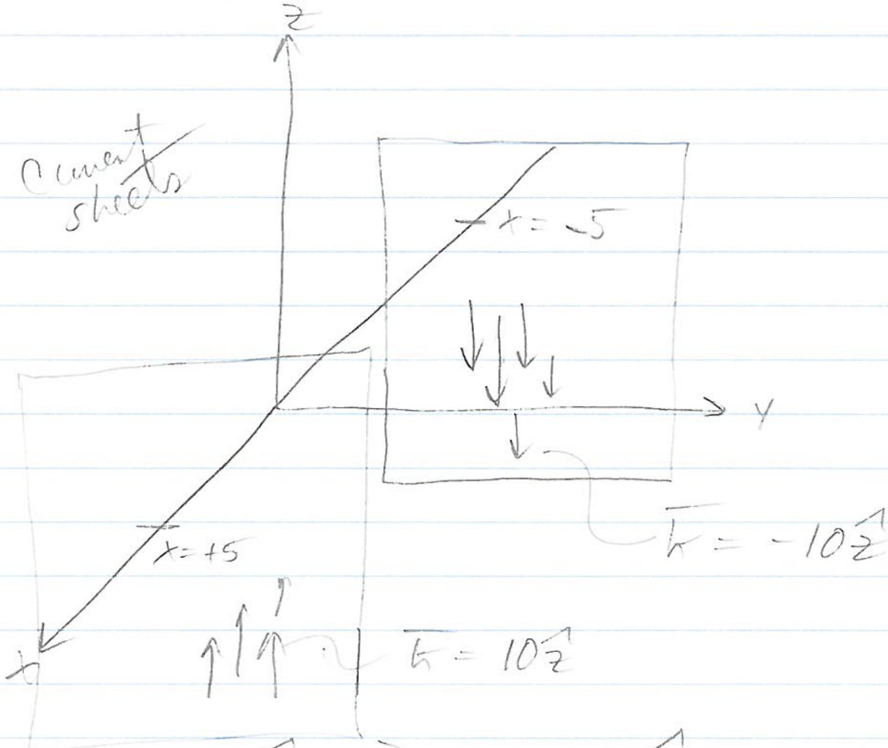
$$d\overline{H} = \frac{I\, d\overline{l} \times \overline{R}}{4\pi R^2}$$

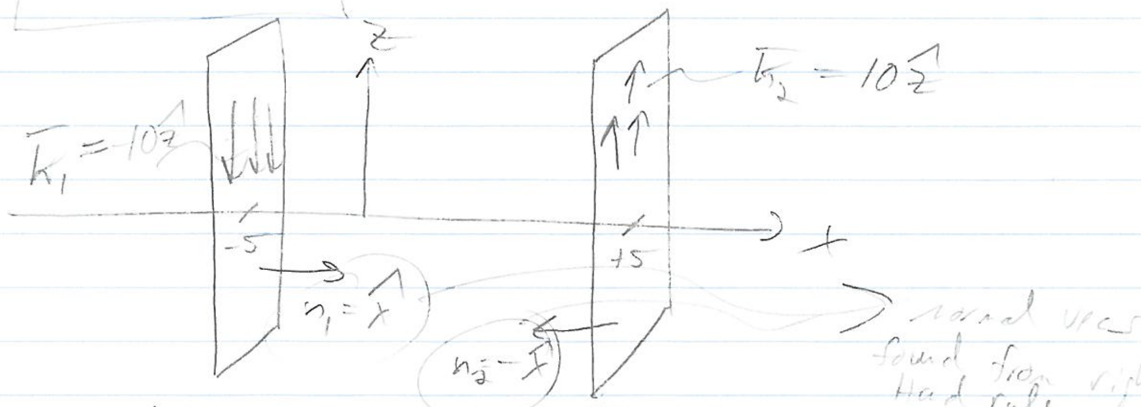
for $0 \le x \le L/2$, $y = -\frac{L}{2}$ is the half side

$$I\,dl = I\hat{x}$$
$$\overline{R} = -x\hat{x} + \frac{L}{2}\hat{y}$$

$$\Rightarrow d\overline{H} = \frac{(I\hat{x}) \times \left(-x\hat{x} + \frac{L}{2}\hat{y}\right)}{4\pi\left(x^2 + \left(\frac{L}{2}\right)^2\right)^{3/2}}$$

9.3) A current sheet, $\bar{K} = 10\hat{z}$ (A/m), lies in the $x = 5$ m plane and a second sheet, $\bar{K} = -10\hat{z}$ (A/m), is at $x = -5$ m. Find $\bar{H}$ at all pts.





— to find H between the plates, sum the two H feilds produced by the plates

$$\bar{H} = \bar{H}_1 + H_2 = -5\hat{y} - 5\hat{y} = \boxed{-10\hat{y} = \bar{H}}$$

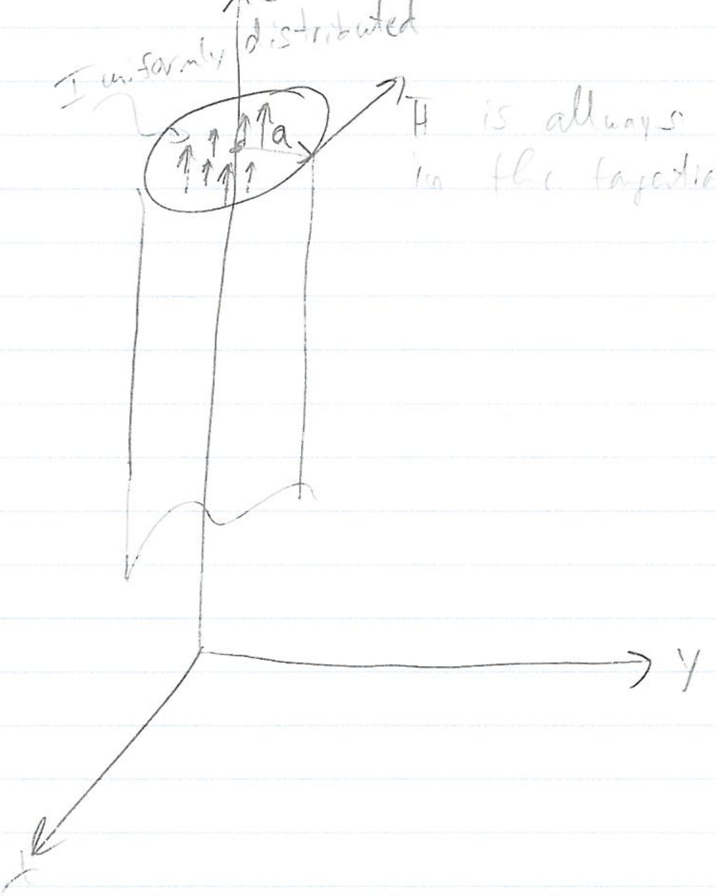
for $-5 < x < 0$, 0 elsewhere

cuz of right hand rule

$$\bar{H}_1 = \frac{1}{2}\bar{K}_1 \times n_1 = -\frac{10}{2}\hat{z} \times \hat{x} = -5\hat{y}$$

$$\bar{H}_2 = \frac{1}{2}K_2 \times n_2 = \frac{10}{2}\hat{z} \times -\hat{x} = -5\hat{y}$$

9.5) Determine $\bar{H}$ for a solid cylindrical conductor of radius a, where the current I is uniformly distributed over the cross section.



- use Ampere's Law to find $\bar{H}$

$$\oint \bar{H} \cdot d\bar{l} = I_{enc}$$

$$d\bar{l} = \phi \rho \, d\phi$$

$$\oint H \phi \rho \, d\phi = I_{enc}$$

$$I_{enc} = \frac{\text{SA of current r pos}}{\text{total SA of wire}}$$

$$2\pi\rho H = I_{enc}, \quad I_{enc} = I\left(\frac{\pi\rho^2}{\pi a^2}\right)$$

so $$2\pi\rho H = I\frac{\pi\rho^2}{\pi a^2}$$

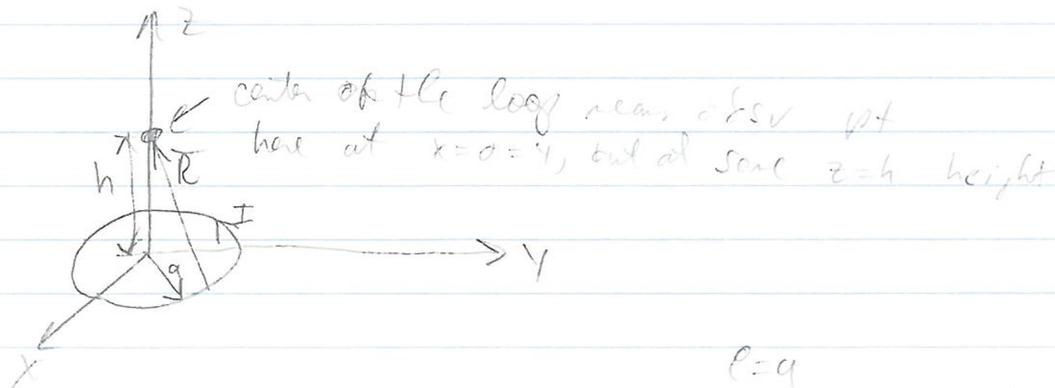
$$\Rightarrow \quad H = \frac{I \pi \rho^2}{\pi a^2} \frac{1}{2\pi\rho}$$

$$H = \frac{I\rho}{2\pi a^2}$$

- H is allways in the tangential direction, in this case it is $\hat{\phi}$
- thus

$$\boxed{\bar{H} = \frac{I\rho}{2\pi a^2}\hat{\phi}}$$

9.7) Find $\bar{H}$ on the axis of a circular current loop of radius a. Specialize the result to the center of the loop:



$$\bar{R} = -a\hat{\rho} + h\hat{z}\ , \quad d\bar{l} = \phi \overset{\rho=a}{\cancel{\rho}} d\phi\hat{\phi} = \phi a\, d\phi\,\hat{\phi}$$

- use Biot-Savart's

$$dH = \frac{I\,d\bar{l} \times \hat{a}_R}{4\pi R^2} = \frac{(I a\, d\phi\, \hat{\phi}) \times (-a\hat{\rho} + h\hat{z})}{4\pi\,(a^2 + z^2)^{3/2}}$$

$$\overset{a}{(I a\, d\phi)} \times \overset{b}{(-a\hat{\rho} + h\hat{z})}$$

$$= \begin{vmatrix} \hat\rho & \hat\phi & \hat z \\ 0 & I a\, d\phi & 0 \\ -a & 0 & h \end{vmatrix} = \hat\rho \begin{vmatrix} I a\, d\phi & 0 \\ 0 & h \end{vmatrix} - \hat\phi \overset{0}{\cancel{\begin{vmatrix} 0 & 0 \\ -a & h \end{vmatrix}}} + \hat z \begin{vmatrix} 0 & I a\, d\phi \\ -a & 0 \end{vmatrix}$$

$$= \hat\rho I a h\, d\phi + \hat z I a^2 d\phi$$

$$= I a\, d\phi (h\hat\rho + a\hat z)$$

$$\Rightarrow dH = \frac{I a\, d\phi (h\hat\rho + a\hat z)}{4\pi (a^2 + z^2)^{3/2}}$$

$$\bar H = \oint \frac{I a\, d\phi (h\hat\rho + a\hat z)}{4\pi (a^2 + z^2)^{3/2}}$$

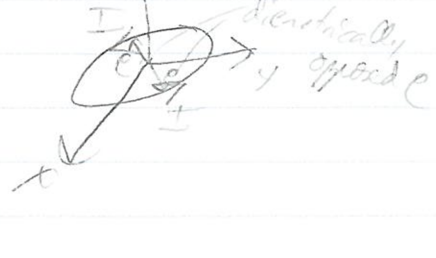
$$= \hat\rho \oint \frac{I a h\, d\phi}{4\pi (a^2 + z^2)^{3/2}} + \hat z \oint \frac{I a^2 d\phi}{4\pi (a^2 + z^2)^{3/2}}$$

$$= \hat\rho \frac{I a h}{2 (a^2 + z^2)^{3/2}} + \hat z \frac{I a^2}{2 (a^2 + z^2)^{3/2}}$$

$$\bar H = \frac{I a}{2 (a^2 + z^2)^{3/2}} \left[\hat\rho h + \hat z a \right]$$

- but, inspection shows that diametrically opposit current elements produce ρ components which cancel, so

$$\boxed{\bar H = \frac{I a^2}{2 (a^2 + z^2)^{3/2}} \hat z}$$



ad finally, at $h = 0$ (center of the circle)

$$\bar{H} = \frac{I a^2}{2(a^2 + 0)^{3/2}} \hat{z} = \boxed{\frac{I}{2a} \hat{z} = \bar{H}}$$

at center

9.9) Given $\bar{A} = (y \cos ax)\hat{x} + (y + e^x)\hat{z}$, find $\nabla \times \bar{A}$ at the origin.

$$\text{Curl of } \bar{A} = \nabla \times \bar{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{x}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \hat{y}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \hat{z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] \quad (1)$$

$$\frac{\partial A_z}{\partial y} = \frac{\partial}{\partial y}\left[y + e^x\right] = 1$$

$$\frac{\partial A_y}{\partial z} = 0$$

$$\frac{\partial A_x}{\partial z} = \frac{\partial}{\partial z}\left[y \cos(ax)\right] = 0,$$

$$\frac{\partial A_z}{\partial x} = \frac{\partial}{\partial z}\left[y + e^x\right] = e^x$$

$$\frac{\partial A_y}{\partial x} = 0$$

$$\frac{\partial A_x}{\partial y} = \frac{\partial}{\partial y}\left[y \cos ax\right] = \cos ax$$

- apply the above results back into (1)

$$= \hat{x}\left[1 - 0\right]$$

$$+ \hat{y}\left[0 - e^x\right]$$

$$+ \hat{z}\left[0 - \cos ax\right]$$

$$= \boxed{\hat{x} - e^x \hat{y} - \cos(ax)\,\hat{z} = \nabla \times \bar{A}}$$ doc

- find $\nabla \times \bar{A}$ at the origin $(0,0,0)$

$$\nabla \times \bar{A}(0,0,0) = \boxed{\hat{x} - \hat{y} - \hat{z}}$$ doc

9.11) Given the general vector field $A = 5\rho \sin\phi\,\hat{z}$ in cylindrical coords, find curl $\bar{A}$ at $(2,\pi,0)$

$$\text{curl } \bar{A} = \nabla \times \bar{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$= \hat{\rho}\begin{vmatrix} \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho A_\phi & A_z \end{vmatrix} - \rho\hat{\phi}\begin{vmatrix} \frac{\partial}{\partial \rho} & \frac{\partial}{\partial z} \\ A_\rho & A_z \end{vmatrix} + \hat{z}\begin{vmatrix} \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} \\ A_\rho & \rho A_\phi \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\hat{\rho}\left[\frac{\partial A_z}{\partial \phi} - \cancelto{0}{\frac{\partial(\rho A_\phi)}{\partial z}}\right] - \rho\hat{\phi}\left[\frac{\partial A_z}{\partial \rho} - \cancelto{0}{\frac{\partial A_\rho}{\partial z}}\right] + \hat{z}\left[\cancelto{0}{\frac{\partial(\rho A_\phi)}{\partial \rho}} - \cancelto{0}{\frac{\partial A_\rho}{\partial \phi}}\right]\right]$$

$$\frac{\partial A_z}{\partial \phi} = \frac{\partial}{\partial \phi}(5\rho\sin\phi) = 5\rho\cos\phi$$

$$\frac{\partial A_z}{\partial \rho} = \frac{\partial}{\partial \rho}(5\rho\sin\phi) = 5\sin\phi$$

$$= \frac{1}{\rho}\left[\hat{\rho}\,5\rho\cos\phi - \hat{\phi}\,5\rho\sin\phi\right] = \hat{\rho}\,5\cos\phi - \hat{\phi}\,5\sin\phi = \nabla\times\bar{A}$$

$$\nabla\times\bar{A}\Big|_{(2,\pi,0)} = \hat{\rho}\,5\cos(\pi) - \hat{\phi}\,5\sin(\pi)$$

$$= \boxed{-\hat{\rho}\,5} \text{ doc}$$

9.13) Given the general vector field $\bar{A} = 10\sin\theta\,\hat{\theta}$ in spherical coords, find $\nabla\times\bar{A}$ at $(2, \frac{\pi}{2}, 0)$:

$$\nabla\times\bar{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{r}\begin{bmatrix}\frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ rA_\theta & r\sin\theta A_\phi\end{bmatrix} - r\hat{\theta}\begin{bmatrix}\frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} \\ A_r & r\sin\theta A_\phi\end{bmatrix} + r\sin\theta\hat{\phi}\begin{bmatrix}\frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} \\ A_r & rA_\theta\end{bmatrix}\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{r}\left[\cancelto{0}{\frac{\partial}{\partial\theta}(r\sin\theta A_\phi)} - \cancelto{0}{\frac{\partial}{\partial\phi}(rA_\theta)}\right] - r\hat{\theta}\left[\frac{\partial}{\partial r}(r\sin\theta A_\theta) - \cancelto{0}{\frac{\partial}{\partial\phi}A_r}\right] + r\sin\theta\hat{\phi}\left[\frac{\partial}{\partial r}(rA_\theta) - \cancelto{0}{\frac{\partial}{\partial\theta}A_r}\right]\right]$$

$$\frac{\partial}{\partial r}\left(r\sin\theta\,(10\sin\theta)\right) = \frac{\partial}{\partial r}\left(10r\sin^2\theta\right) = 10\sin^2\theta$$

$$\frac{\partial}{\partial r}\left(r\,10\sin\theta\right) = 10\sin\theta$$

$$= \frac{1}{r^2\sin\theta}\left[-\hat{\theta}\,10r\sin^2\theta + r\sin\theta\,\hat{\phi}\,10\sin\theta\right]$$

$$= \frac{1}{r^2\cancel{\sin\theta}}\left[-10r\sin^2\theta\,\hat{\theta} + 10r\sin^2\theta\,\hat{\phi}\right]$$

$$= \left.\frac{-10\sin\theta}{r}\hat{\theta} + \frac{10\sin\theta}{r}\hat{\phi}\right|_{(2,\frac{\pi}{2},0)} = \boxed{\frac{-10\sin\frac{\pi}{2}}{2}\hat{\theta} + \frac{10\sin\frac{\pi}{2}}{2}\hat{\phi}}$$

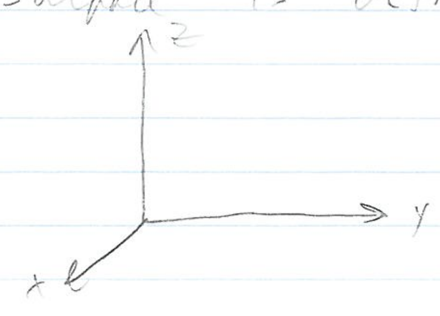
9.15) A radial field $\bar{H} = \frac{2.39E6}{r}\cos\phi\,\hat{r}$ (A/m) exists in free space. Find the magnetic flux Φ crossing the surface defined by $-\pi/4 \le \phi \le \pi/4$, $0 \le z \le 1$m.

- use the magnetic flux formula

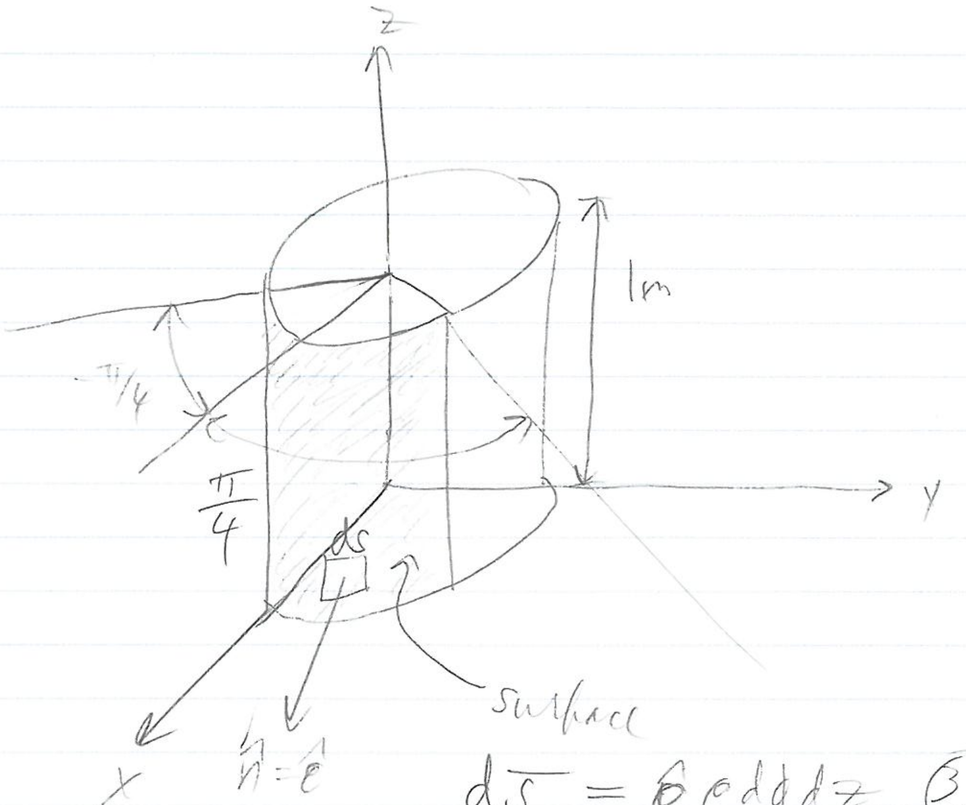
$$\Phi = \int_S \bar{B}\cdot d\bar{S} \quad (1)$$

- where $\bar{B} = \mu\bar{H} = \mu_0\bar{H} = \frac{\mu_0\, 2.39E6}{\rho}\cos\phi\,\hat{\rho} = \bar{B}$ (2)

- the surface is defined in cylindrical coords by



over



$d\bar{S} = \hat{\rho}\rho \, d\phi \, dz$ ③

so we use dS_ρ

- apply 2 and 3 to 1

$$\Phi = \int_0^1 \int_{-\frac{\pi}{4}}^{\pi/4} \frac{\mu_0 \, 2.39E6}{\cancel{\rho}} \cos\phi \, \hat{\rho} \cdot \hat{\rho} \, \cancel{\rho} \, d\phi \, dz$$

$$= \mu_0 \, 2.39E6 \int_0^1 \int_{-\frac{\pi}{4}}^{\pi/4} \cos\phi \, d\phi \, dz$$

$4\pi E\text{-}7$

$$= \cancel{\mu_0} \, 2.39E6 \int_0^1 \left[-\sin\phi \right]_{-\frac{\pi}{4}}^{\pi/4} dz = 3 \int_0^1 \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right] dz$$

$$= 3\sqrt{2} = \boxed{\Phi = 4.24 \text{ wb}}$$

9.17) Obtain the vector magnetic potential $\bar{A}$ in the region surrounding an infinitely long, straight, filamentary current I:

- use the vector magnetic potential $\bar{A}$ eq

$$\nabla \times \bar{A} = \bar{B}$$

$$\bar{B} = \mu_0 \bar{H} = \frac{\mu_0 I}{2\pi \rho} \hat{\phi}$$

$$\nabla \times \bar{A} = \frac{\mu_0 I}{2\pi \rho} \hat{\phi}$$

$$\nabla \times \bar{A} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \frac{1}{\rho}\left[\cancel{\hat{\rho} \begin{bmatrix} \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho A_\phi & A_z \end{bmatrix}}^{0} - \rho\hat{\phi} \begin{bmatrix} \frac{\partial}{\partial \rho} & \frac{\partial}{\partial z} \\ A_\rho & A_z \end{bmatrix} + \cancel{\hat{z} \begin{bmatrix} \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} \\ A_\rho & \rho A_\phi \end{bmatrix}}^{0} \right]$$

$$\nabla \times \bar{A} = -\hat{\phi}\left[\frac{\partial A_z}{\partial \rho} - \frac{\partial A_\rho}{\partial z}\right]$$

$$\nabla \times \bar{A} = -\hat{\phi}\frac{\partial A_z}{\partial \rho} + \hat{\phi}\cancel{\frac{\partial A_\rho}{\partial z}}^{0} = \frac{\mu_0 I}{2\pi \rho}\hat{\phi}$$

Since the infinite line has a const current and is infinite on z, this term can be thrown out

$$-\hat{\phi}\frac{\partial A_z}{\partial \rho} = \frac{\mu_0 I}{2\pi \rho}\hat{\phi} \qquad (1)$$

— solve DE 1

$$\int \frac{\partial A_z}{\partial \rho} d\rho = \int \frac{-\mu_0 I}{2\pi \rho} d\rho$$

$$A_z = \frac{-\mu_0 I}{2\pi} \ln \rho + C \quad (2)$$

- the constant of integration permits the location of a zero reference (needed when finding the potential of anything).

- set $A_z = 0$ at $\rho = \rho_0$

$$0 = \frac{-\mu_0 I}{2\pi} \ln \rho_0 + C$$

$$\Rightarrow C = \frac{\mu_0 I}{2\pi} \ln \rho_0 \quad (3)$$

- apply 3 back into 2

$$A_z = \frac{-\mu_0 I}{2\pi} \ln \rho + \frac{\mu_0 I}{2\pi} \ln \rho_0$$

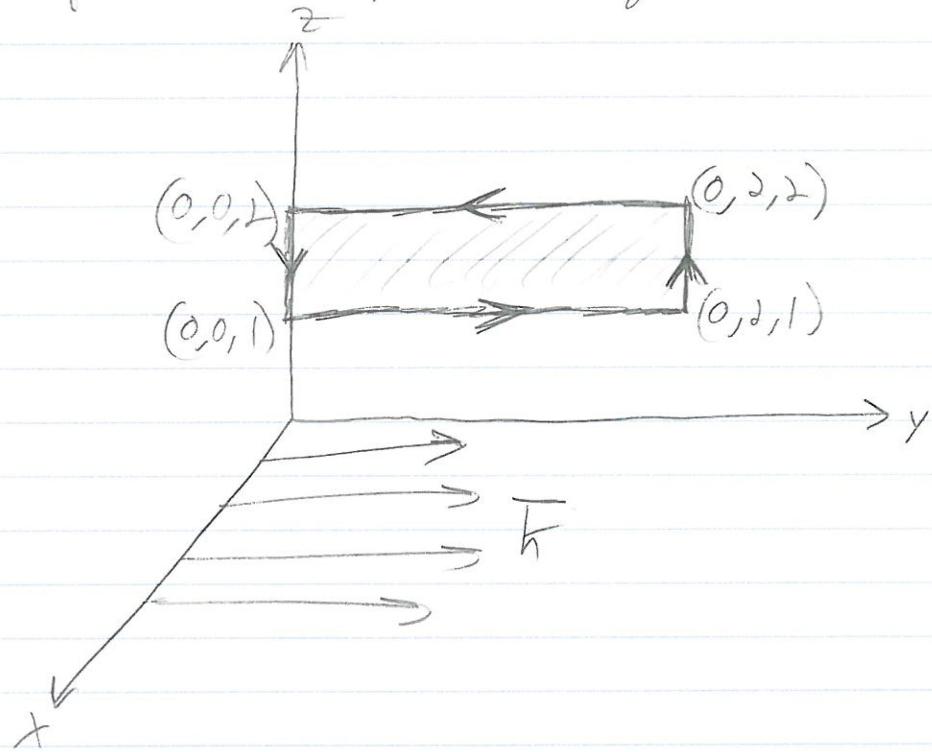
$$A_z = \frac{\mu_0 I}{2\pi} \ln\left(\frac{\rho_0}{\rho}\right)$$

- turn this into a vector

$$\boxed{\bar{A} = \hat{z} \frac{\mu_0 I}{2\pi} \ln\left(\frac{\rho_0}{\rho}\right)}$$

done

9.19) Given $\bar{A} = -\frac{\mu_0}{2}(z - z_0)\bar{k}$, find the magnetic flux crossing the rectangular area:



- Since we are finding the mag. potential, we need to assign a zero reference pt.
  - in this case, we will put the zero reference at $z_0 = 2$, making

$$\bar{A} = -\frac{\mu_0}{2}(z - 2)\bar{k}$$

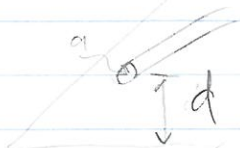
- use the mag flux integral

$$\Phi = \int_S \bar{B} \cdot d\bar{S} = \oint \bar{A} \cdot d\bar{l}$$

by Stoke's theorem

11.3) A circular conductor, with radius a = 0.803", is located 12.5 ft from an infinite conducting plane. Find the inductance:

- use the standard conductor configuration:



$$\frac{L}{l} = \frac{\mu_0}{2\pi}\cosh^{-1}\left(\frac{d}{a}\right) \left(H/m\right)$$

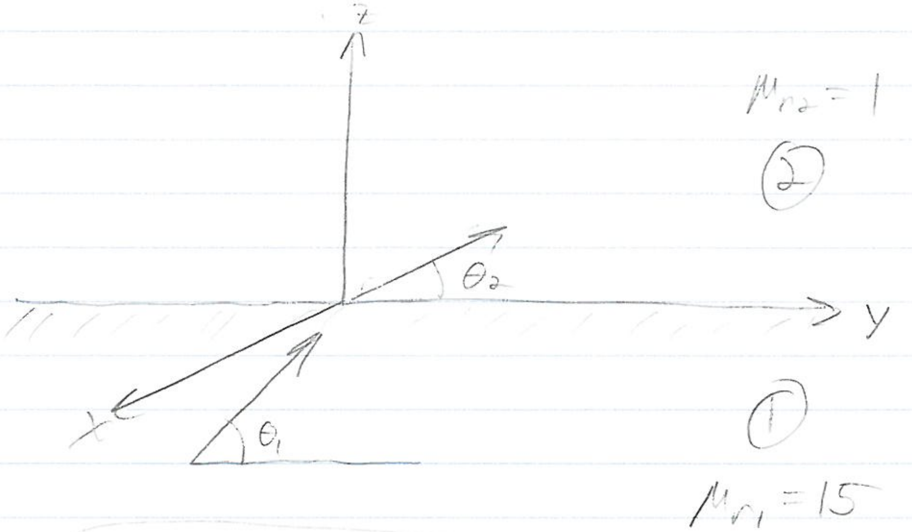
$$= \frac{\mu_0}{2\pi}\cosh^{-1}\left(\frac{12.25}{.803}\right) \quad H/m$$

11.5) Assume the air core inductor in fig 11-3 has 700 turns, an inner radius of 0.01 m, an outer radius of 0.02 m, and a height a = 1.5 cm. Find L using:

a) the formula

Ch 13: Maxwell's Equations and Boundary Equations

13.1) In Region 1, $\bar{B}_1 = 1.2\hat{x} + 0.8\hat{y} + 0.4\hat{z}$ (T). Find $\bar{H}_2$ (ie $\bar{H}$ at $z = 0^+$) and the angles between the field vectors and a tangent to the interface:



$$\bar{B}_1 = 1.2\hat{x} + 0.8\hat{y} + 0.4\hat{z} \qquad ①$$

$$\bar{H}_1 = \frac{1}{\mu_r \mu_0} \bar{B}_1 = \frac{1}{15\mu_0}(1.2\hat{x} + 0.8\hat{y} + 0.4\hat{z})$$

$$\bar{H}_1 = \frac{0.08\hat{x}}{\mu_0} + \frac{0.053\hat{y}}{\mu_0} + \frac{0.026\hat{z}}{\mu_0} \qquad ②$$

- apply boundary conditions

$\rightarrow H_{T1} = H_{T2}$

$$\Rightarrow \bar{H}_2 = \frac{0.8\hat{x}}{\mu_0} + \frac{0.053\hat{y}}{\mu_0} + H_{z2}\hat{z} \qquad ③$$

$\rightarrow B_{n1} = B_{n2}$

$$\Rightarrow \bar{B}_2 = B_{x2}\hat{x} + B_{y2}\hat{y} + 0.4\hat{z} \qquad ④$$

$$B_{x2} = \mu_0 \mu_{r2} \underbrace{H_{x2}}_{H_{x1}} = \mu_0(1)\frac{0.08}{\mu_0} = 0.08 = B_{x2}$$

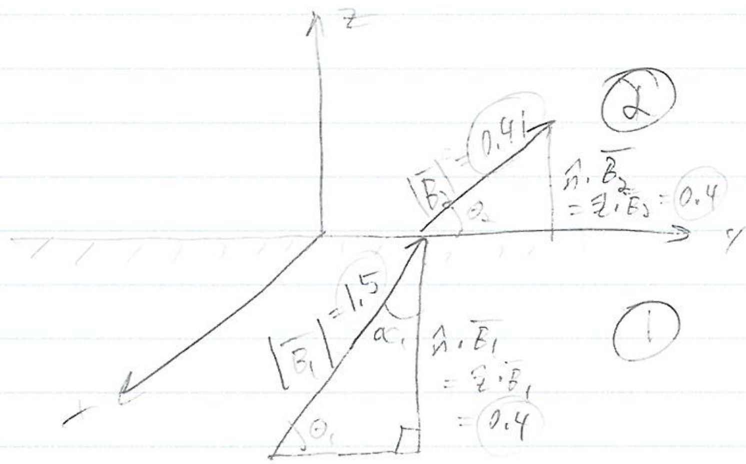
$$B_{y2} = \mu_0 \mu_{r2} \underbrace{H_{y2}}_{H_{y1}} = \mu_0(1)\frac{0.053}{\mu_0} = 0.053 = B_{y2}$$

$$H_{z2} = \frac{\cancel{B_{z2}}^{\,B_{z1}}}{\mu_0\mu_{r2}} = \frac{0.4}{\mu_0(1)} = \frac{0.4}{\mu_0} = H_{z2}$$

thus

$$\boxed{\begin{aligned}\overline{H_2} &= \frac{0.8\hat{x}}{\mu_0} + \frac{0.053\hat{y}}{\mu_0} + \frac{0.4\hat{z}}{\mu_0} \\ \overline{B_2} &= 0.08\hat{x} + 0.053\hat{y} + 0.4\hat{z}\end{aligned}}$$

- next, find the angles θ_1 and θ_2



$\sin = \frac{opp}{hyp}$

$\cos = \frac{adj}{hyp}$

$$|\overline{B_1}| = \sqrt{1.2^2 + 0.8^2 + 0.4^2} = 1.5$$

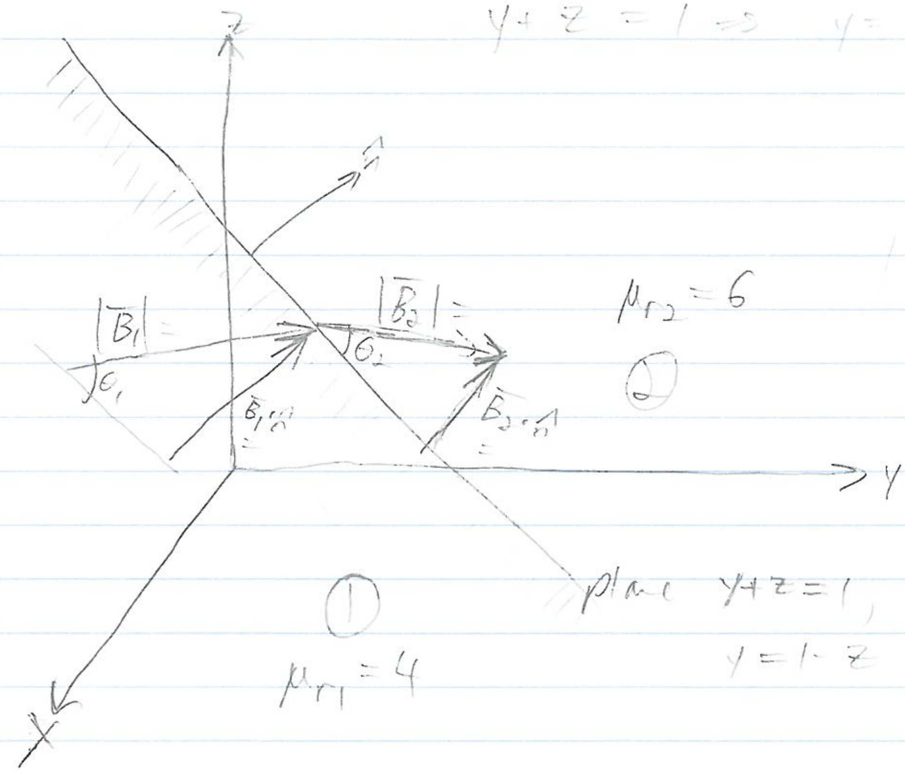
$$\sin\theta_1 = \frac{0.4}{1.5} \Rightarrow \boxed{\theta_1 = 15.5^\circ}$$

$$|\overline{B_2}| = \sqrt{0.08^2 + 0.053^2 + 0.4^2} = 0.41$$

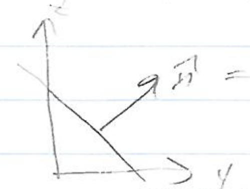
$$\sin\theta_2 = \frac{0.4}{0.41} \Rightarrow \boxed{\theta_2 = 76.71^\circ}$$

13.3) Region 1, where $\mu_{r_1} = 4$, is the side of the plane $y + z = 1$ containing the origin. In region 2, $\mu_{r_2} = 6$. $\bar{B}_1 = 2\hat{x} + 1\hat{y}$ (T), find $\bar{B}_2$ and $\bar{H}_2$:

$y + z = 1 \Rightarrow y = 1 - z$



- find the normal (unit normal vector) $\hat{n}$



$$\hat{n} = \frac{\hat{y} + \hat{z}}{\sqrt{1^2 + 1^2}} = \frac{\hat{y}}{\sqrt{2}} + \frac{\hat{z}}{\sqrt{2}} = \hat{n} \quad (1)$$

$$\bar{B}_1 = 2\hat{x} + \hat{y} \quad (2)$$

$$\bar{H}_1 = \frac{1}{\mu_0 \mu_{r_1}} \bar{B}_1 = \frac{0.5\hat{x}}{\mu_0} + \frac{0.25\hat{y}}{\mu_0} = \bar{H}_1 \quad (3)$$

- apply the boundary condition $B_{n_1} = B_{n_2}$
- in order to apply this boundary condition, we must first identify B_{n_1}, B_{n_2}

over →

$B_{n1} = \bar{B_1} \cdot \hat{n} = (2\hat{x} + \hat{y}) \cdot \left(\frac{\hat{y}}{\sqrt{2}} + \frac{\hat{z}}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}}$

$B_{n1} = \frac{1}{\sqrt{2}}$

$\bar{B_{n1}} = (B_{n1})\hat{n} = \frac{1}{\sqrt{2}} \left(\frac{\hat{y}}{\sqrt{2}} + \frac{\hat{z}}{\sqrt{2}} \right) = \frac{\hat{y}}{2} + \frac{\hat{z}}{2} = \bar{B_{n1}}$ (4)

- apply boundary condition

$\Rightarrow \bar{B_{n1}} = \bar{B_{n2}}$

$\Rightarrow \bar{B_{n2}} = \frac{\hat{y}}{2} + \frac{\hat{z}}{2}$ (5)

- in order to find $\bar{H_{T1}}$, we must find $\bar{B_{T1}}$

$\bar{B_{T1}} = \bar{B} - \bar{B_{n1}} = (2\hat{x} + \hat{y}) - (0.5\hat{y} + 0.5\hat{z})$

$\bar{B_{T1}} = 1.5\hat{x} + \hat{y} - 0.5\hat{z}$

- using the above equation, we find $\bar{H_{T1}}$

$\bar{H_{T1}} = \frac{1}{\mu_0 \mu_{r1}} \bar{B_{T1}} = \frac{1}{\mu_0 4} (1.5\hat{x} + \hat{y} - 0.5\hat{z})$

$\bar{H_{T1}} = \frac{0.375}{\mu_0}\hat{x} + \frac{0.25}{\mu_0}\hat{y} - \frac{0.125}{\mu_0}\hat{z}$ (6)

* apply the boundary condition *

$\Rightarrow \bar{H_{T1}} = \bar{H_{T2}} = \frac{0.375}{\mu_0}\hat{x} + \frac{0.25}{\mu_0}\hat{y} - \frac{0.125}{\mu_0}\hat{z}$ (7)

- in order to finish the problem, we need to find $\bar{B_{T2}}$

over →

$$\overline{B}_{T2} = \mu_0\mu_{r2}\,\overline{H}_{T2} = \mu_0(6)\left[\frac{0.375}{\mu_0}\hat{x} + \frac{0.25}{\mu_0}\hat{y} - \frac{0.125}{\mu_0}\hat{z}\right]$$

$$\overline{B}_{T2} = 2.25\,\hat{x} + 1.5\,\hat{y} - 0.75\,\hat{z}$$

- and finally, we find $\overline{B}_2$ by adding $\overline{B}_{T2} + \overline{B}_{N2}$

$$\overline{B}_2 = \overline{B}_{N2} + \overline{B}_{T2} = (0.5\,\hat{y} + 0.5\,\hat{z}) + (2.25\,\hat{x} + 1.5\,\hat{y} - 0.75\,\hat{z})$$

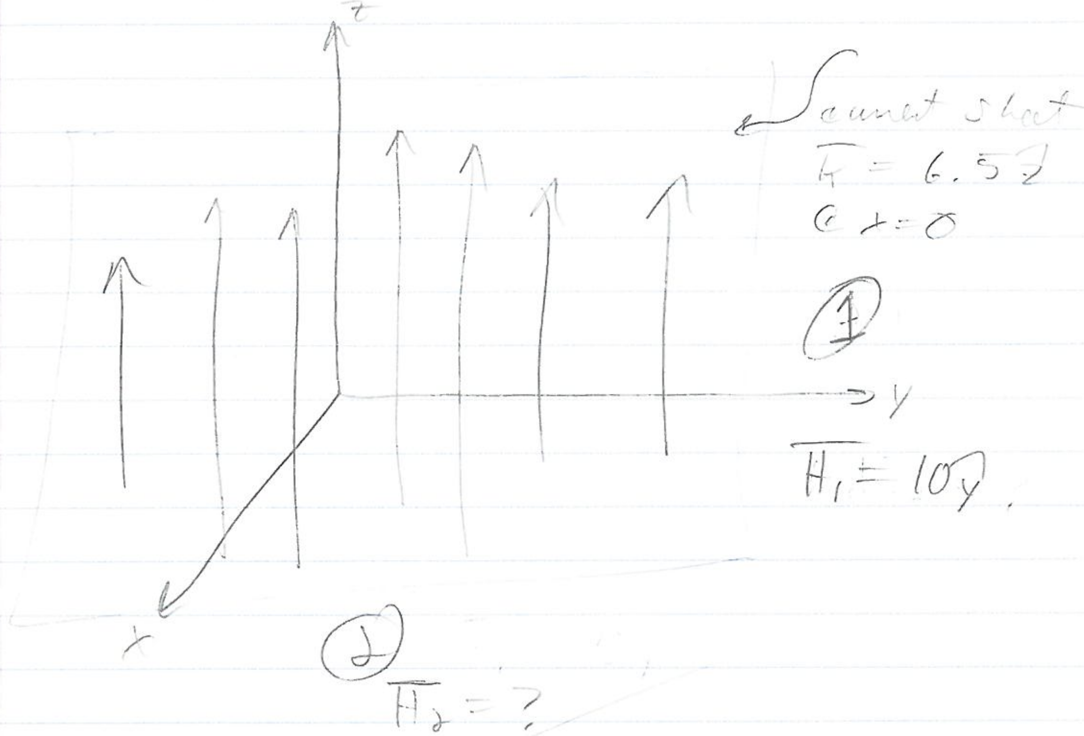
$$\boxed{\overline{B}_2 = 2.25\,\hat{x} + 2\,\hat{y} - 0.25\,\hat{z}}$$ done

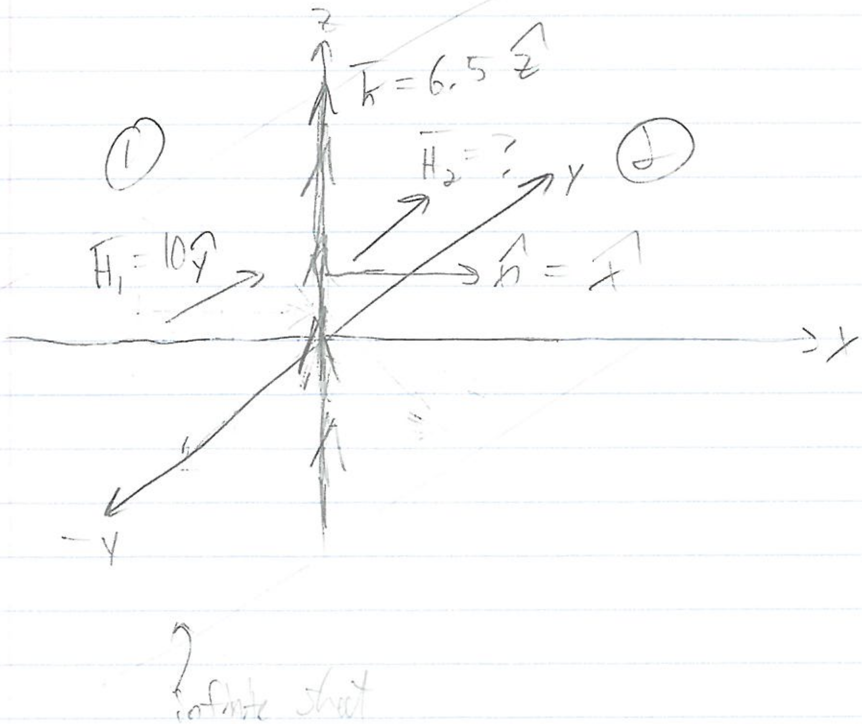
- finally, we find $\overline{H}_2$ from $\overline{B}_2$

$$\overline{H}_2 = \frac{1}{\mu_0\mu_{r2}}\overline{B}_2 = \frac{1}{\mu_0(6)}\left[2.25\,\hat{x} + 2\,\hat{y} - 0.25\,\hat{z}\right]$$

$$\boxed{\overline{H}_2 = \frac{0.375}{\mu_0}\hat{x} + \frac{0.333}{\mu_0}\hat{y} - \frac{0.042}{\mu_0}\hat{z}}$$ done

13.5) A current sheet, $\bar{K} = 6.5\hat{z}$ (A/m), at $x=0$ separates region 1, $x<0$, where $\bar{H}_1 = 10\hat{y}$ (A/m) and region 2, $x>0$. Find $\bar{H}_2$ at $x=0^+$.





$\bar{B}_1 = \mu_0 \mu_r \bar{H}_1 = \mu_0 (10\hat{y})$

$\bar{B}_1 = \mu_0 10\hat{y}$

- apply the boundary condition $B_{n1} = B_{n2}$

$B_{n1} = 0 = B_{n2}$ since there are no normal components of B_1

- apply the other boundary condition:
  - note: in this case there is a current sheet on the boundary, so we have to use the other form of the 2nd boundary cond:

$$(\bar{H}_1 - \bar{H}_2) \times \hat{n}_{12} = \bar{K}$$

$$(10\hat{y} - H_{y2}\hat{y}) \times \hat{x} = 6.5\hat{z}$$

$$(10 - H_{y2})\,\hat{y} \times \hat{x} = 6.5\hat{z},$$

$$\hat{y} \times \hat{x} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \hat{x}\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - \hat{y}\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + \hat{z}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -\hat{z}$$

$$(10 - H_{y2})(-\hat{z}) = 6.5\hat{z}$$

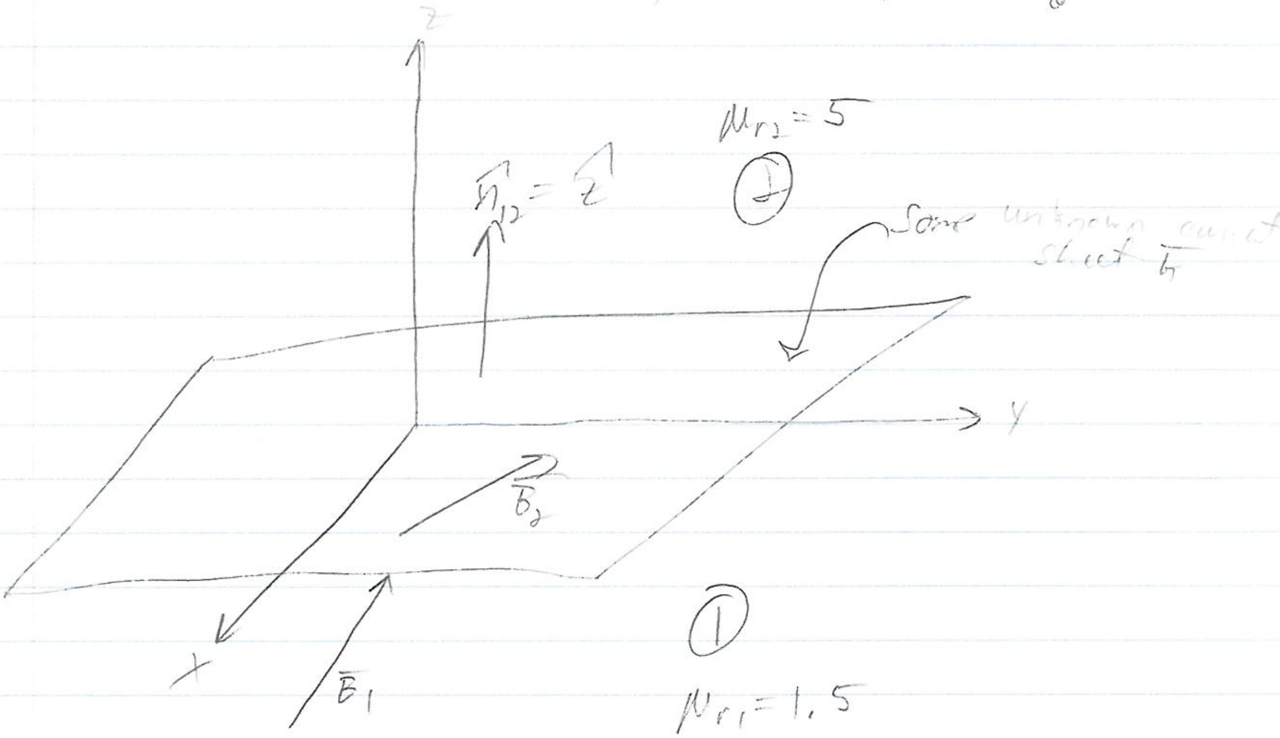
$$-(10 - H_{y2}) = 6.5 \Rightarrow H_{y2} = 10 + 6.5 = 16.5 = H_{y2}$$

$\Rightarrow \boxed{\bar{H}_2 = 16.5\hat{y}}$ done

13.7) Region 1, $z < 0$, has $\mu_{r1} = 1.5$, while region 2, $z > 0$, has $\mu_{r2} = 5$. Near $(0,0,0)$,

$$\bar{B}_1 = 2.4\hat{x} + 10\hat{z}, \quad \bar{B}_2 = 25.75\hat{x} - 17.7\hat{y} + 10\hat{z}$$

If the interface carries a sheet current, what is its density at the origin?



- in order to solve for $\bar{K}$, we will use the 2nd boundary condition in the presence of a current sheet

$$(\bar{H}_1 - \bar{H}_2) \times \hat{n}_{12} = \bar{K} \qquad \text{eq 1a}$$

- find $\bar{H}_1$

$$\bar{H}_1 = \frac{1}{\mu_0 \mu_{r1}} \bar{B}_1 = \frac{1}{1.5\mu_0}(2.4\hat{x} + 10\hat{z}) = \boxed{\frac{1.6}{\mu_0}\hat{x} + \frac{6.67}{\mu_0}\hat{z} = \bar{H}_1}$$

eq 1

- apply eq. 4 back into eq 3

$$\boxed{\frac{1}{\mu_0}\left(3.54\hat{x} + 3.55\hat{y}\right)}$$ done

13.9) a) Show that the E and H fields of problem 13.8 constitute a wave traveling in the z direction.
b) Verify that the wave speed and $\frac{E}{H}$ depend only on the properties of free space.

- from problem 13.8

$$\bar{E} = E_m \sin(\omega t - \beta z)\hat{y}$$

$$\bar{H} = -\frac{\beta E_m}{\omega \mu_0} \sin(\omega t - \beta z)\hat{x}$$

a) Since E and H vary with space and time through the sinusoidal eq $\sin(\omega t - \beta z)$, then the wave is a traveling wave

let

$$\omega t - \beta z = \text{const} = \omega t_0$$

$$\Rightarrow z = \frac{\omega(t - t_0)}{\beta}$$, this is the equation

- for a plane moving with speed $c = \frac{\omega}{\beta}$ in the direction of its normal $\hat{z}$,

- thus, the entire ~~pattern~~ wave moves down the z axis with speed c

13.11) Given

$$\bar{E} = 30\pi e^{j(10^8 t + \beta z)} \hat{x} \quad (V/m)$$

$$\bar{H} = H_m e^{j(10^8 t + \beta z)} \hat{y} \quad (A/m)$$

in free space, find H_m and β $(\beta > 0)$

- use the traveling wave relations

$$\frac{\omega}{\beta} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c = 3E8\,(m/s)$$

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \;(\Omega) = \frac{30\pi e^{j(10^8 t + \beta z)}}{H_m e^{j(10^8 t + \beta z)}}$$

- apply maxwell's eq $\nabla \times E = -\frac{\partial B}{\partial t}$

$$\nabla \times E = \hat{y}\left(\frac{\partial E_x}{\partial z}\right) = \hat{y}\, 30\pi\beta e^{j(10^8 t + \beta z)}$$

$$-\frac{\partial B}{\partial t} = -\hat{y} H_m 10^8 e^{j(10^8 t + \beta z)}$$

$$\hat{y}\, 30\pi\beta e^{j(10^8 t + \beta z)} = -\hat{y} H_m 10^8 e^{j(10^8 t + \beta z)}$$

$$\Rightarrow H_m = \frac{-30\pi\beta \cancel{e^{j(10^8 t + \beta z)}}}{10^8 \cancel{e^{j(10^8 t + \beta z)}}}$$

$$\boxed{H_m = \frac{-30\beta\pi}{10^8}}$$ doe

Ch 14: Electromagnetic Waves

14.1) A traveling wave is described by $y = 10\sin(\beta z - \omega t)$. Sketch the wave at $t=0$ and at $t = t_1$, when it has advanced $\lambda/8$ if the velocity is 3E8 m/s and the angular frequency $\omega = 10^6$ rad/s. Repeat for $\omega = 2E6$ rad/s and the same t_1.

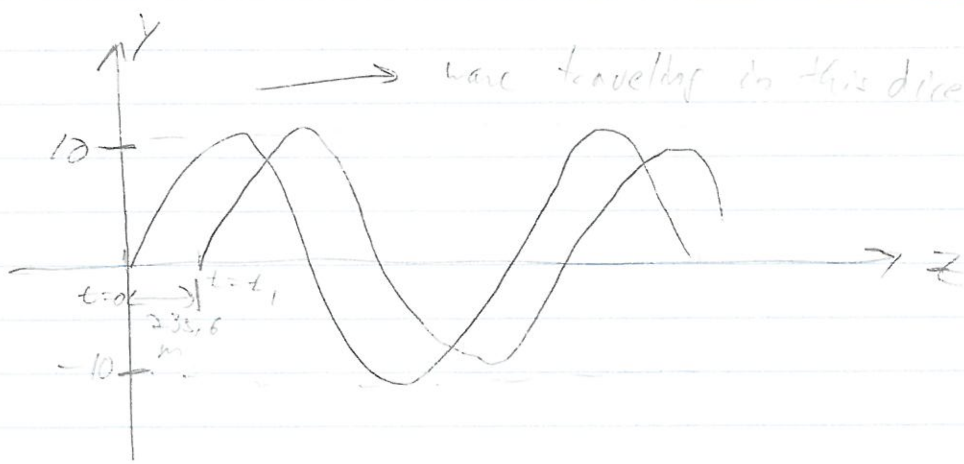
$$y = 10\sin(\beta z - \omega t), \quad c = 3E8$$

amplitude along the y axis — traveling in the z direction

→ for $\omega = 10^6$ rad/s

$$\lambda = \frac{c}{f} = \frac{2\pi c}{\omega} = \frac{2\pi\, 3E8}{10^6} = 1{,}885 \text{ m}$$

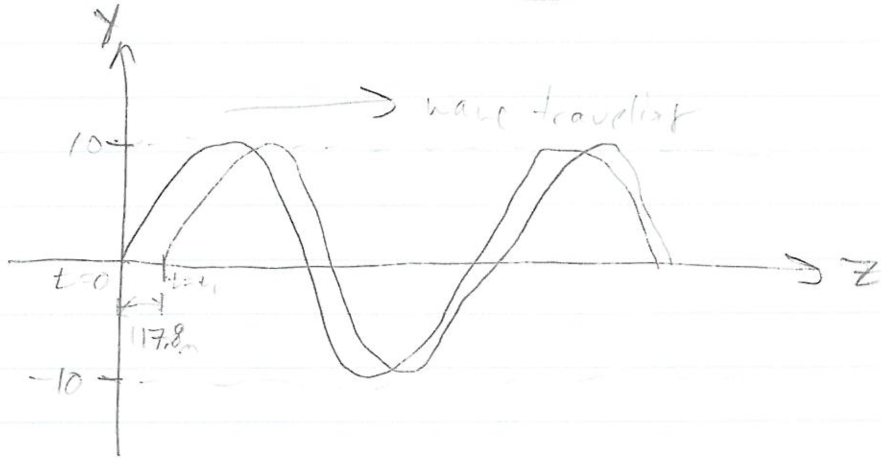
$$\frac{\lambda}{8} = \frac{1885}{8} = 235.6 \text{ m}$$ is how far the wave has advanced



→ for $w = 2E6$

$$\lambda = \frac{2\pi c}{w} = \frac{2\pi\, 3E8}{2E6} = 942.5$$

$$\frac{\lambda}{8} = \frac{942.5}{8} = 117.8\,m$$



14.3) For the wave problem in 14.2, determine the propagation constant γ, given that the frequency $f = 95.5\,MHz$

- from the vector wave equations

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$\Rightarrow \gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}, \quad \omega = 2\pi f = 2\pi\, 95.5E6$$
$$\omega = 600E6$$

- since we are in free space, the conductivity $\sigma = 0$, $\varepsilon = \varepsilon_0 = 8.854E-12$, $\mu = \mu_0 = 12.566E-7$

$$\gamma = \sqrt{j(600E6)(12.566E-7)(\varnothing + j(600E6)(8.854E-12))}$$

$$\gamma = \sqrt{-4} = \boxed{j2\, m^{-1} = \gamma}$$

14.5) An $\bar{H}$ field travels in the $-\hat{z}$ direction in free space with a phase shift constant of 30.0 rad/m and an amplitude of $(1/3\pi)$ (A/m). If the field has the direction $-\hat{y}$ when $t=0$ and $z=0$, write suitable expressions for $\bar{E}$ and $\bar{H}$. Determine the frequency and wavelength.

$$\bar{H}(z,t) = H_o e^{\pm \gamma z} e^{j\omega t} \hat{a}_H$$

$$\gamma = \alpha + j\beta$$

H field varies along z

field travelling in this direction

$$\bar{H}(z,t) = -H_o e^{+z(\alpha+j\beta)} e^{j\omega t} \hat{y} = -H_o e^{(z(\alpha+j\beta)+j\omega t)} \hat{y}$$

$$= -H_o e^{z\alpha + j\beta z + j\omega t} \hat{y} = -H_o e^{z\alpha + j(\beta z + \omega t)} \hat{y}$$

$$= -H_o e^{z\alpha} e^{j(\beta z + \omega t)} \hat{y}$$

$$= -H_o e^{z\alpha} \left[\cos(\beta z + \omega t) + j\sin(\beta z + \omega t)\right] \hat{y}$$

$$H_o = \frac{1}{3\pi}, \quad \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1-\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)}$$

- Since we are in free space, $\sigma = 0$, thus

$$\alpha = \omega\sqrt{\frac{\mu_o\varepsilon_o}{2}\left(\sqrt{1-0^2} - 1\right)}$$

$$\alpha = 0$$

$$\Rightarrow \bar{H} = -\frac{1}{3\pi} \cancel{e^{z(0)}} \left[\cos(\beta z + \omega t) + j\sin(\beta z + \omega t)\right] \hat{y}$$

$$\boxed{\bar{H} = -\frac{1}{3\pi}\left[\cos(\beta z + \omega t) + j\sin(\beta z + \omega t)\right] \hat{y}} \quad (1)$$

- Find the intrinsic impedance η

$$\eta = \sqrt{\frac{j\omega\mu}{\cancel{\sigma} + j\omega\varepsilon}} = \frac{E_{a_E}}{H_{a_H}} = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi = \frac{E_{a_E}}{H_{a_H}} \quad (2)$$

from ②, we know that both E and H are real, thus we can ignore the imaginary comp's of each

$$\boxed{\overline{H} = \frac{-1}{3\pi}\cos(\beta z+\omega t)\hat{y}} \quad ③$$

we use eq ① for this

$$(\sigma + j\omega\varepsilon)\overline{E} = \nabla \times \overline{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \hat{x}\left[\cancel{\frac{\partial H_z}{\partial y}}^{0} - \frac{\partial H_y}{\partial z}\right] + \hat{y}\left[\cancel{\frac{\partial H_x}{\partial z}}^{0} - \cancel{\frac{\partial H_z}{\partial x}}^{0}\right] + \hat{z}\left[\cancel{\frac{\partial H_y}{\partial x}}^{0} - \cancel{\frac{\partial H_x}{\partial y}}^{0}\right]$$

$$(\sigma + j\omega\varepsilon)\overline{E} = -\hat{x}\frac{\partial H_y}{\partial z} = \frac{+\hat{x}}{3\pi}\frac{\partial}{\partial z}\left[\cos(\beta z+\omega t) + j\sin(\beta z+\omega t)\right]$$

$$= \frac{\hat{x}}{3\pi}\left[-\beta\sin(\beta z+\omega t) + j\beta\cos(\beta z+\omega t)\right]$$

$$(\sigma + j\omega\varepsilon)\overline{E} = \frac{\hat{x}\beta}{3\pi}\left[-\sin(\beta z+\omega t) + j\cos(\beta z+\omega t)\right]$$

$$E = \frac{\hat{x}\beta}{3\pi(\sigma + j\omega\varepsilon)}\left[-\sin(\beta z+\omega t) + j\cos(\beta z+\omega t)\right] \longrightarrow$$

---

from ② and ③

$$120\pi = \frac{E_x}{\frac{-1}{3\pi}\cos(\beta z+\omega t)} \Rightarrow E_x = \frac{-120}{3\pi}\cos(\beta z+\omega t)$$

$$\rightarrow \boxed{\overline{E} = \frac{-120}{3\pi}\cos(\beta z+\omega t)\hat{x}}$$

$$\rightarrow E = \hat{x}\left[\frac{-\beta}{3\pi(\cancel{\sigma}+j\omega\varepsilon)}\sin(\beta z+\omega t) + \frac{j\beta}{3\pi(\cancel{\sigma}+j\omega\varepsilon)}\cos(\beta z+\omega t)\right]$$

$\sigma = 0$

$$E = \hat{x}\left[\frac{-\beta}{j3\pi\omega\varepsilon}\sin(\beta z+\omega t) + \frac{\cancel{j}\beta}{\cancel{j}3\pi\omega\varepsilon}\cos(\beta z+\omega t)\right]$$

$$E = \hat{x}\left[\frac{j\beta}{3\pi\omega\varepsilon}\sin(\beta z+\omega t) + \frac{\beta}{3\pi\omega\varepsilon}\cos(\beta z+\omega t)\right]$$

Since we are in free space, we can ignore the imaginary part

$$E = \frac{\hat{x}\beta}{3\pi\omega\varepsilon}\cos(\beta z+\omega t) = \boxed{39.95\cos(30z+9E7t)\hat{x} = E}$$

done

$$\frac{\beta}{3\pi\omega\varepsilon} = \frac{30}{3\pi(9E7)(8.854E-12)} = 39.95$$

$\varepsilon = \varepsilon_0 = 8.854E-12$

given $\beta = 30 \Rightarrow \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{30} = \frac{\pi}{15}$

$\lambda = \frac{c}{f} \Rightarrow f = \frac{c}{\lambda} = \frac{3E8}{\pi/15} = 1.43E7\ Hz \Rightarrow \omega = 2\pi f = 9E7$

14.9) Find the skin depth δ at a frequency of 1.6 MHz in aluminum, where $\sigma = 38.2$ MS/m and $\mu_r = 1$. Also find γ and the wave velocity u.

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi (1.6E6)(12.566E-7)(38.2E6)}}$$

$\boxed{\delta = 64.4\ \mu m}$ is the skin depth

– in good conductors, $\sigma >> \omega\varepsilon$, therefore

$$\alpha = \beta = \delta^{-1} = \frac{1}{64.4\mu} = 15.53E3$$

$$\Rightarrow \gamma = \alpha + j\beta = \boxed{15.53E3 + j15.53E3 = \gamma}$$ is the prop. const.

– for the wave velocity u

$$u = \frac{\omega}{\beta} = \frac{2\pi\, 1.6E6}{15.53E3} = \boxed{647.3\ m/s = u}$$ is the wave velocity

14.11) Calculate the intrinsic impedance η, the propagation constant γ, and the wave velocity u for a conducting medium in which $\sigma = 58$ MS/m, $\mu_r = 1$, at a freq of $f = 100$ MHz:

- for the intrinsic impedance η

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ = \sqrt{\frac{(2\pi 100E6)(12.566E-7)}{58E6}} \angle 45^\circ$$

$$\boxed{\eta = 3.69E-3 \angle 45^\circ}$$

- for the propagation constant γ

$$\gamma = \alpha + j\beta$$

for a good conductor

$$\alpha = \beta = \sqrt{\pi f \mu \sigma} = \sqrt{\pi(100E6)(12.566E-7)(58E6)} = 151.3E3$$

$$\Rightarrow \boxed{\gamma = 151.3E3 + j151.3E3}$$

- for the wave velocity u

$$u = \frac{\omega}{\beta} = \frac{2\pi 100E6}{151.3E3} = \boxed{4.15E3 \text{ m/s} = u}$$

14.10) A perpendicularly polarized wave propagates from region 1 ($\varepsilon_{r1} = 8.5$, $\mu_{r1} = 1$, $\sigma_1 = 0$) to region 2, free space, with an angle of incidence of 15°. Given $E_o^i = 1.0\ \mu V/m$, find: E_o^r, E_o^t, H_o^r, and H_o^t.

- first, find the intrinsic impedances

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{1(12.566E-7)}{8.5(8.854E-12)}} = 129.2$$

$$\eta_2 = \eta_0 = 120\pi$$

- next, use Snell's law to find the angles

$$\boxed{\theta_i = \theta_r = 15^\circ}$$

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}} \Rightarrow \frac{\sin 15^\circ}{\sin\theta_t} = \sqrt{\frac{(12.566E-7)(8.854E-12)}{(12.566E-7)(8.5)(8.854E-12)}}$$

$$\Rightarrow \sin\theta_t = 0.754 \Rightarrow \boxed{\theta_t = 48.9^\circ}$$

- finally, find the various amplitudes

$$\frac{E_o^r}{E_o^i} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{120\pi\cos 15^\circ - 129.2\cos 48.9^\circ}{120\pi\cos 15^\circ + 129.2\cos 48.9^\circ}$$

$$\frac{E_o^r}{E_o^i} = 0.622$$

$$\frac{E_0^t}{E_0^i} = \frac{2\eta_2 \cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{2(120\pi)\cos(15^\circ)}{120\pi\cos(15^\circ) + 127.2\cos(48.9^\circ)}$$

$$\frac{E_0^t}{E_0^i} = 1.62$$

since $\boxed{E_0^i = 1.0E-6}$ $\Rightarrow \boxed{\begin{array}{l} E_0^t = 1.62E-6 \\ \Rightarrow E_0^r = 0.622E-6 \end{array}}$

$$H_0^i = \frac{E_0^i}{\eta_1} = \frac{1\mu}{129.2} = \boxed{7.74E-9 = H_0^i}$$

$$H_0^t = \frac{E_0^t}{\eta_2} = \frac{1.26E-6}{120\pi} = \boxed{3.34E-9}$$

$$H_0^r = \frac{E_0^r}{\eta_1} = \frac{0.622E-6}{129.2} = \boxed{4.81E-9}$$

14.13) In free space $E(z,t) = 50\cos(\omega t - \beta z)\hat{x}$ (V/m). Find the average power crossing a circular area of radius 2.5m in the plane z = const.

- convert E to complex form

$$\overline{E} = 50 e^{j(\omega t - \beta z)} \hat{x}$$

- Since we are in free space, $\eta = \eta_0 = 120\pi$, and since propagation is in the +z direction:

$$\overline{H} = \frac{50}{120\pi} e^{j(\omega t - \beta z)} \hat{y}$$

- use the Poynting vector to find the average power

$$P_{avg} = \frac{1}{2} Re\left(\overline{E} \times \overline{H}^*\right) = \frac{1}{2}(50)\left(\frac{50}{120\pi}\right)\hat{z}$$

$$P_{avg} = 3.316\,\hat{z} \quad W/m^2$$

- Since the circular area is located on the z axis, the power going through that circular area is integrated

$$P_{avg} = (3.316)(\pi 2.5^2) = \boxed{65.1\,W}$$

14.15) Determine the amplitudes of the reflected and transmitted $\bar{E}$ and $\bar{H}$ at the interface shown in fig 14-12, if $E_0^i = 1.5E-3$ V/m in region 1, in which $\varepsilon_{r1} = 8.5$, $\mu_{r1} = 1$, and $\sigma = 0$. Region 2 is free space. Assume normal incidence

$$\eta_1 = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{12.566E-7}{8.5(8.854E-12)}} = 129.2$$

$$\eta_2 = \eta_0 = 120\pi$$

$$E_0^i = 1.5E-3$$

$$\frac{E_0^r}{E_0^i} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} = \frac{120\pi - 129.2}{129.2 + 120\pi} = 0.489$$

$$\frac{E_0^t}{E_0^i} = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{2(120\pi)}{120\pi + 129.2} = 1.5$$

$$\frac{H_0^r}{H_0^i} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} = \frac{129.2 - 120\pi}{129.2 + 120\pi} = -0.489$$

$$\frac{H_0^t}{H_0^i} = \frac{2\eta_1}{\eta_1 + \eta_2} = \frac{2(129.2)}{129.2 + 120\pi} = 0.510$$

$$\Rightarrow E_0^r = 0.489(1.5E-3) = \boxed{733.5E-6}$$

$$E_0^t = 1.5(1.5E-3) = \boxed{2.25E-3}$$

$$H_0^i = \frac{E_0^i}{\eta_1} = \frac{1.5E-3}{129.2} = \boxed{11.6E-6}$$

$$H_0^r = -0.487(11.6E-6) = \boxed{-5.67E-6}$$

$$H_0^t = 0.510(11.6E-6) = \boxed{5.92E-6}$$

4.17) A normally incident E field has amplitude $E_0^i = 1.0$ (V/m) in free space just outside of seawater in which $\varepsilon_r = 80$, $\mu_r = 1$, $\sigma = 2.5$ (S/m). For a frequency of 30 MHz, at what depth will the amplitude of E be 1.0E-3 (V/m)?

- let the free space region be 1, and seawater be 2

$$\eta_1 = 120\pi, \quad \eta_2 = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j(2\pi 30E6)(12.566E-7)}{2.5 + j(2\pi 30E6)80(8.854E-12)}}$$

$\omega = 2\pi 30E6$

$$\Rightarrow \boxed{\eta_2 = 9.73 \angle 45^\circ}$$

- then, the amplitude of E just inside of the seawater is

$$\frac{E_2^t}{E_0^i} = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{2(9.73\angle 45^\circ)}{120\pi + 9.73\angle 45^\circ} = (0.0507 \angle 44^\circ)$$

$$\Rightarrow E_2^t = (0.0507\angle 44^\circ)\,1 = \boxed{(0.0507 \angle 44^\circ) \text{ V/m} = E_0^t}$$

- find the propagation constant γ

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \sqrt{j(2\pi 30E6)(12.566E-7)(2.5 + j(2\pi 30E6)(80)(8.854E-12))}$$

$$\gamma = 24.35 \angle 46.52 = 16.75 + j17.67 = \alpha + j\beta$$

$$\Rightarrow \alpha = 16.75$$

- using the general form of the traveling wave

$$\bar{E} = E_0 e^{-\gamma z} \hat{x}$$

$$E = E_0 e^{-\gamma z}$$

apply E_0 results from above:

$\cancel{*} \longrightarrow E_0 = E_0^{t} = 5E-2, \; E = 1mV$

ignore the complex part

$$1E-3 = 5E-2 e^{-(\alpha z + j\beta z)}$$

$$1E-3 = 5E-2 e^{-\alpha z} = e^{-16.75 z}$$

$$\Rightarrow \boxed{z = 0.233m} \text{ dist}$$

Ch 15: Transmission Lines

15.1) A parallel-wire transmission line is constructed of #6 AWG copper wire (dia = 0.162 in, σ_c = 58 MS/m) with a 12 inch separation in air. Neglecting internal inductance, find the per-meter values of L, C, G, the dc resistance, and ac resistance at 1 MHz:

radius $a = 2.06E-3$ m
diameter $d = 0.305$ m

$$GFC = \frac{1}{GFL} = \frac{1}{5} = 0.2$$

$$GFL = \ln\frac{d}{a} = \ln\left(\frac{0.305}{2.06E-3}\right) = 5$$

$$GFR_d = \frac{2}{a^2} = \frac{2}{(2.06E-3)^2} = 471.3E3$$

$$GFR_a = \frac{2}{a} = \frac{2}{2.06E-3} = 970.9$$

$$C = \pi\varepsilon_d(GFC) = \pi(8.854E-12)(0.2) = \boxed{5.56\ pF/m}$$

$$G = \frac{C}{\varepsilon_d}\sigma_d = \frac{5.56p}{(8.854E-12)}(0) = \boxed{0}$$

$$L_e = \frac{\mu_d}{\pi}(GFL) = \frac{(12.566E-7)}{\pi}(5) = \boxed{2\ \mu H/m}$$

$$R_d = \frac{1}{\sigma_c\pi}(GFR_d) = \frac{1}{58E6\,\pi}(471.3E3) = \boxed{2.68\ m\Omega/m}$$

$$R_a = \frac{1}{2\pi\sigma_c\delta}(GFR_a) = \frac{(970.9)}{2\pi(58E6)(66E-6)} = \boxed{40.4\ m\Omega/m}$$

$$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}} = \frac{1}{\sqrt{\pi(1E6)(12.566E-7)(58M)}} = 66\ \mu m$$

$$L_i = \frac{R_a}{2\pi f} = \frac{40.4E-3}{2\pi\ 1E6} = 6.43E-9\ H/m$$

15.3) Show that the voltage $v(x,t) = A\cos(\omega t + \theta)e^{j\beta x}$ satisfies the transmission line equation (3), for a uniform lossless line, if $\beta = \omega\sqrt{LC}$:

- transmission line equation (eq 3 in book)

$$\frac{\partial^2 f(x,t)}{\partial x^2} = RGf(x,t) + (RC+LG)\frac{\partial f(x,t)}{\partial t} + LC\frac{\partial^2 f(x,t)}{\partial t^2}$$

- assume a lossless line $\Rightarrow R = G = 0$
- apply the $v(x,t)$ equation

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \frac{LC\,\partial^2 v(x,t)}{\partial t^2}$$

$$\frac{\partial^2}{\partial x^2}\left[A\cos(\omega t+\theta)e^{j\beta x}\right] = LC\frac{\partial^2}{\partial t^2}\left[A\cos(\omega t+\theta)e^{j\beta x}\right]$$

$$\frac{\partial}{\partial x}\left[+j\beta A\cos(\omega t+\theta)e^{j\beta x}\right] = LC\frac{\partial}{\partial t}\left[-\omega A\sin(\omega t+\theta)e^{j\beta x}\right]$$

$$-\beta^2 A\cos(\omega t+\theta)e^{j\beta x} = LC\left[-\omega^2 A\cos(\omega t+\theta)e^{j\beta x}\right]$$

$$\boxed{-\beta^2 A\cos(\omega t+\theta)e^{j\beta x} = -LC\omega^2 A\cos(\omega t+\theta)e^{j\beta x}}$$

- in order for this to be true

$$\beta^2 = LC\omega^2 \Rightarrow \boxed{\beta = \omega\sqrt{LC}}$$

15.5) a 10 km parallel-wire line operating at 100 kHz has $z_0 = 557\,\Omega$, $\alpha = 2.4E{-}5$ Np/m, and $\beta = 2.12E{-}3$ rad/m. For a matched termination at $x=0$, and $\hat{V}_R = 10\angle 0°$ V, evaluate $\hat{V}(x)$ at x increments of $\lambda/4$ and plot the phasors:

$$V(x) = V^+ e^{\gamma x} + V^- e^{-\gamma x} = \hat{V}_{inc}(x) + \hat{V}_{refl}(x)$$

Since the T-line is fed into a matched load, there is no reflected power, thus:

$$V(x) = V^+ e^{\gamma x} = 10\angle 0° e^{\gamma x}$$

$$\hat{V}(x) = 10 e^{(\alpha + j\beta)x} = 10 e^{\alpha x} e^{j\beta x} = 10 e^{\alpha x} \angle \beta x = \hat{V}(x)$$

$$\Rightarrow \boxed{\hat{V}(x) = 10 e^{2.4E-5x} \angle (2.12E{-}3\, x)}$$

$\lambda = 3E8/100k = 3E3 \Rightarrow \frac{\lambda}{4} = \frac{3E3}{4} = 750\text{m}$

$x = n750$

$x = \frac{n\lambda}{4}$

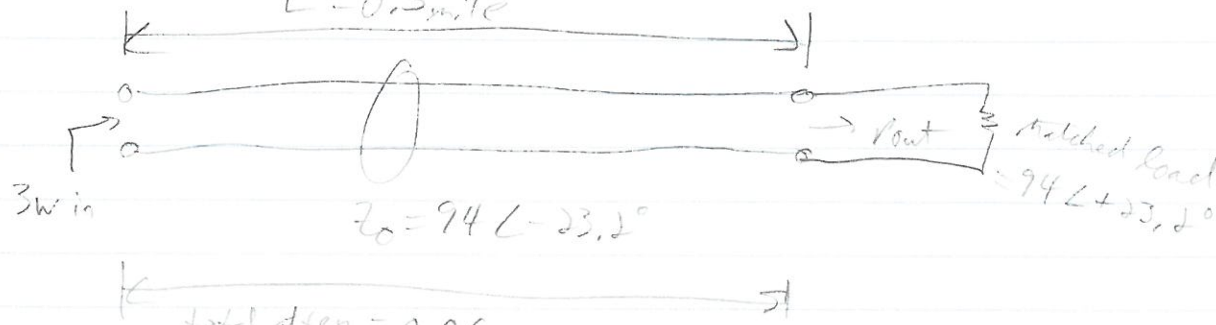
n	$\hat{V}(x)$
0	$10\angle 0°$
1	$10.2\angle 1.6°$
2	$10.4\angle 3.2°$
3	$10.55\angle 4.77°$
4	$10.74\angle 6.36°$
5	$10.94\angle 7.95°$
6	$11.14\angle 9.54°$
7	⋮
8	⋮
9	⋮
10	

n pos from load

15.7)

$f = 5\,kHz$

$L = 0.5\,mile$



total atten = 0.06
phase shift = 8°

- find R, L, G, and C per mile
- find the phase velocity
- " " power lost on the line when the input power is 3 w.

$$0.06 = \alpha(0.5) \Rightarrow \alpha = 0.12 \ Np/mile$$

$$8° = \beta(0.5) \Rightarrow \beta = 16°/mile$$
$$\Rightarrow \beta = 0.28 \ r/mile$$

$$\gamma = \alpha + j\beta = 0.12 + j0.28 = 0.304\angle 66°$$

$$\gamma = \sqrt{ZY} \Rightarrow 0.304\angle 66° = \sqrt{ZY} \quad (1)$$

$$Z_0 = \sqrt{\frac{Z}{Y}} = (94\angle -23.2°) \quad (2)$$

- using eq's 1 and 2 we solve for Z and Y

$$(0.304\angle 66°)^2 = ZY, \quad (94\angle -23.2°)^2 = \frac{Z}{Y}$$

$$(0.092\angle 132°) = ZY \Rightarrow Z = \frac{(0.092\angle 132°)}{Y}$$

$$(94\angle -23.2°) = \frac{(0.092\angle 132°)}{Y^2}$$

$$\Rightarrow Y^2(94\angle -23.2°) = (0.092\angle 132°)$$

$$Y^2 = \frac{(0.092 \angle 132^\circ)}{(94 \angle -23.2^\circ)} = (978E-6 \angle 155.2^\circ)$$

$$Y = \sqrt{978E-6 \angle 155.2^\circ} = (31.3E-3 \angle 77.6^\circ)$$

$$Y = 6.71E-3 + j30.54E-3 = G + j\omega C$$

$$\Rightarrow \boxed{G = 6.71E-3}$$

$$\overset{5kHz}{\omega} C = 30.54E-3 \Rightarrow \boxed{C = 6.1\,\mu F/\text{mile}}$$

$$(0.092 \angle 132^\circ) = Z(31.3E-3 \angle 77.6^\circ)$$

$$\Rightarrow Z = (2.94 \angle 54.4^\circ) = 1.71 + j2.39$$

$$= R + j\omega L$$

$$\Rightarrow \boxed{R = 1.71\ \Omega/\text{mile}}$$

$$\overset{5E3}{\omega} \cancel{L} = 2.39 \Rightarrow \boxed{L = 428\,\mu H/\text{mile}}$$

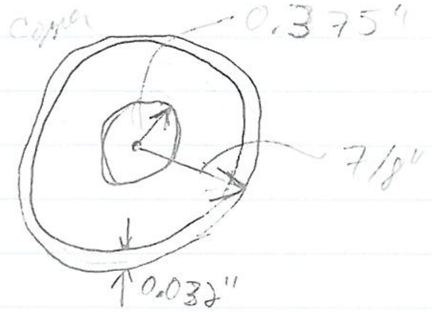
- for the phase velocity

$$u_p = \frac{2\pi f}{\beta} = \frac{2\pi 5E3}{0.28} = \boxed{112.2E3\ mi/s}$$

- for the power recieved by the matched load

$$P_R = P_s e^{-2\alpha l} = 3e^{-2(0.12)(1)} = \boxed{2.4\ W}$$

15.9) For the coaxial line:



$f = 3GHz$

$R_0 = 46.4\ \Omega$, $\alpha = 0.066\ dB/m$

$L = 0.154\ \mu H/m$
$C = 71.9\ pF/m$
$G = 0\ S/m$
$R_a = 0.702\ \Omega/m$

determine the actual characteristic impedance and attenuation, and compare the values with the specifications.

Determine the length of the shorted stub required to support the center conductor at 3GHz and calculate the highest "safe" frequency of operation for this line from the specifications.

– characteristic impedance

$$R_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.154E-6}{71.9E-9}} = \boxed{46.28 = R_0}$$

– attenuation constant

$$\alpha = \frac{R_a}{2R_0} = \frac{0.702}{2(46.28)} = \boxed{7.58E-3\ NP/m}$$

- the stub to support the center conductor at 3GHz must be $\lambda/4$ @ 3GHz

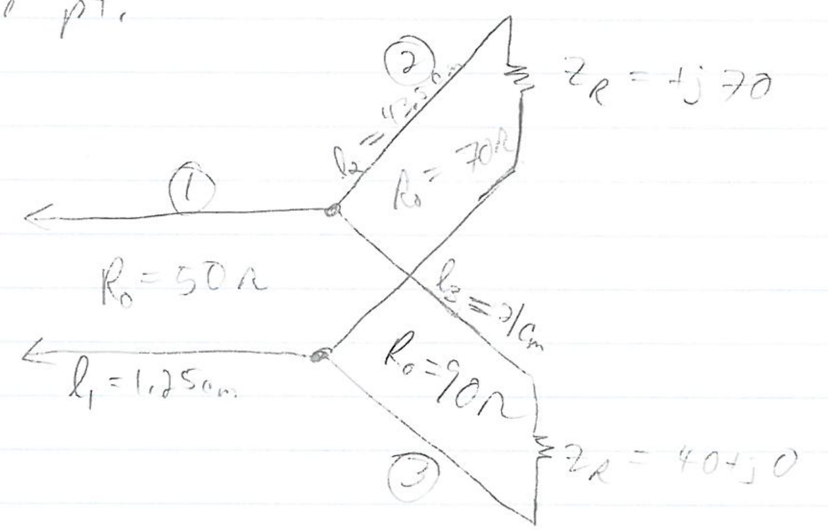
$$\lambda = \frac{3e8}{3GHz} = 0.100\,m$$

$$\frac{\lambda}{4} = \frac{0.1}{4} = 0.025\,m = l_{stub}$$

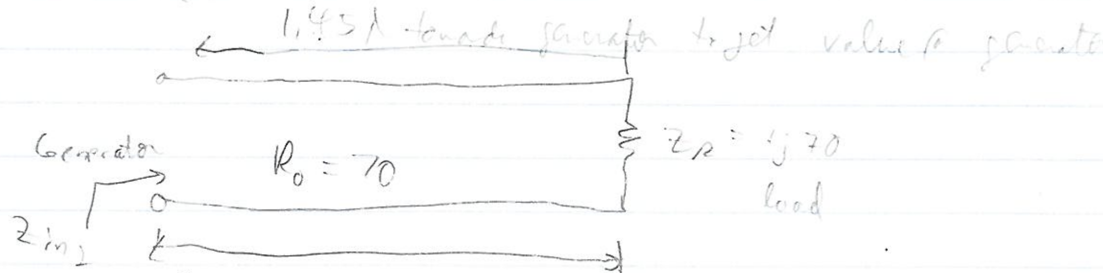
- the "safe" highest freq of operation is determined by the specification for the lowest "safe" freq:

$$f_{hi} = \frac{u_p}{\lambda_{low}} = \frac{3e8}{5.28e-2} = 5.68\,GHz$$

15.11) The high-freq lossless transmission sys. shown in fig 15-22 operates at 700 MHz with a phase velocity for each line of 2.1 E 8 M/S. Use the smith chart to find the VSWR on each section of line and the input impedance to line #1 at the drive pt.



⟹ for line 2



$\ell_2 = 43.5\text{cm} = 1.45\lambda$

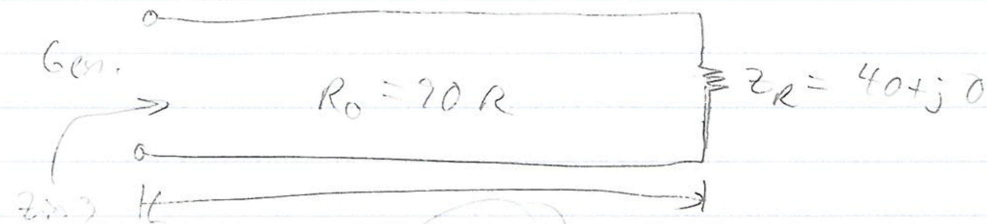
$f = 700\,\text{MHz} \Rightarrow \lambda = \frac{2.1E8}{700E6} = 0.3\,\text{m}$

$\ell_2 = \frac{0.435}{0.3} = 1.45\lambda$ → draw circle from pt 1.45λ towards generator

$z_R = \frac{Z_R}{70} = \frac{j70}{70} = +j$ → plot pt.

$z_{in_2} = j0.51 = 70(j0.51) = \boxed{j35.7 = Z_{in_2}}$

⇒ for line 3



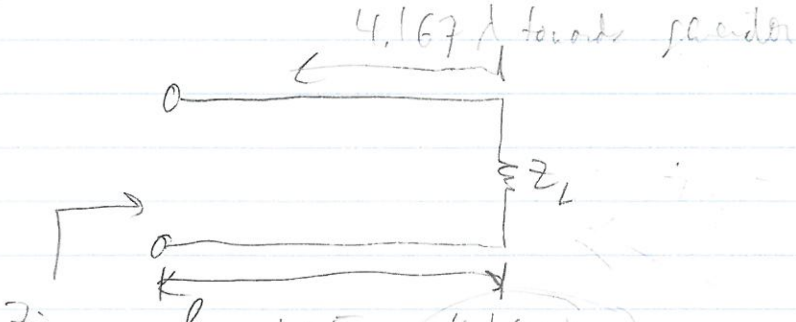
$l_3 = 0.21\,m = 0.7\lambda$

$$l_3 = \frac{0.21}{0.3} = 0.7\lambda$$

$$z_R = \frac{40 + j0}{90} = 0.44 + j0$$

$$Z_{in3} = (1.25 + j0.85)\,90 = \boxed{112.5 + j\,76.5 = Z_{L}}$$

⇒ for line 1



$l_1 = 1.25\,m = 4.167\lambda$

$$l_1 = \frac{1.25\,m}{0.3\,m} = 4.167\lambda$$

$$z_{in2} = \frac{j35.7}{50} = j0.714$$

$$\Rightarrow y_{in2} = \frac{1}{j0.714} = -j1.4$$

$$z_{in_3} = \frac{(112.5 + j76.5)}{50} = 2.25 + j1.53$$

$$Y_{in_3} = \frac{1}{2.25 + j1.53} = 0.304 - j0.207$$

$$Y_L = Y_{in_2} + Y_{in_3} = (-j1.4) + (0.304 - j0.207)$$

$$Y_L = 0.304 - j1.193$$

$$\Rightarrow z_L = \frac{1}{Y_L} = \frac{1}{0.304 - j1.193} = 0.201 + j0.787 = z_L$$

$$z_{in_1} = (3.4 - j4)50 = \boxed{170 - j200 = z_{in_1}}$$ done

$$\boxed{SWR = 8}$$

chart for 15.11

The Complete Smith Chart

Black Magic Design

for line 2

+j0.51 @ 1.575λ

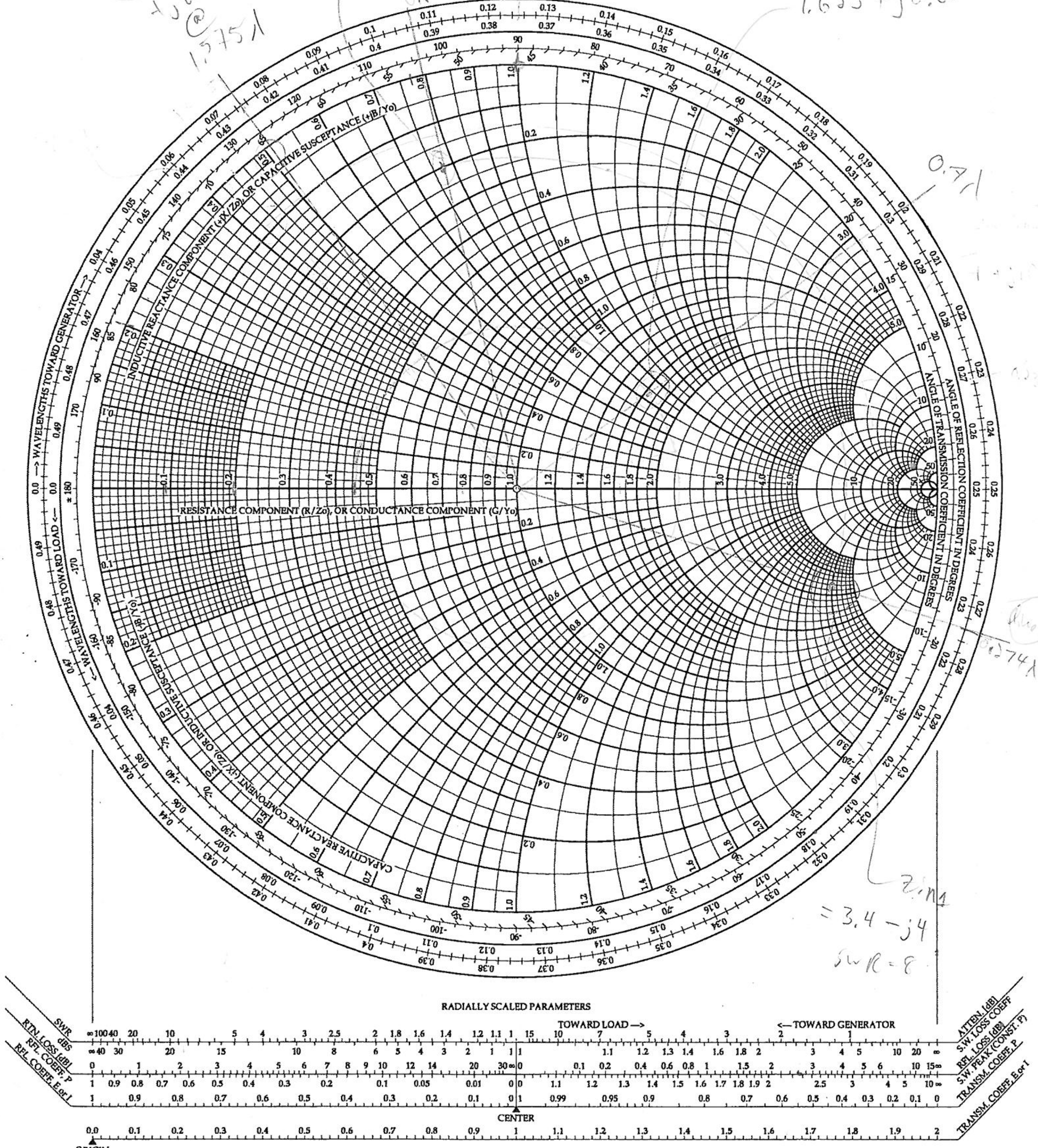
ZL1

0.107

0.125 + j

for line 3

1.625 + j0.85



0.7λ

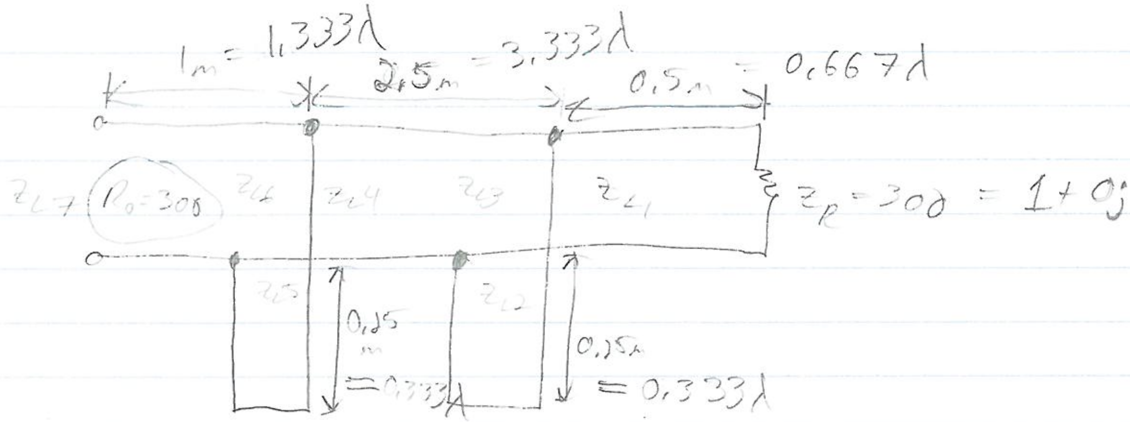
0.274λ

Zin1 = 3.4 − j4

SWR = 8

11

15.13)



f : 400MHz

$$\lambda = \frac{3E8}{400E6} = 0.75\,m$$

$$z_R = \frac{300 + 0j}{300} = 1 + 0j$$

- use a ∥ Y chart side instead of z because all the stubs are in parallel

$$Y_R = \frac{1}{z_R} = 1 + 0j$$

$$Y_{L1} = Y_R = 1 + 0j$$

$$Y_{L2} = Y_{L5} = 0 + j0.575$$

$$Y_{L3} = Y_{L2} + Y_{L1} = 1 + j0.575$$

$$Y_{L4} = 0.625 - j0.09$$

$$Y_{L6} = Y_{L5} + Y_{L4} = j0.575 + 0.625 - j0.09$$

$$Y_{L6} = 0.625 + j0.485$$

$$Y_{L7} = 0.66 - j0.4 \Rightarrow z_{L7} = 300(1.108 + j0.671)$$

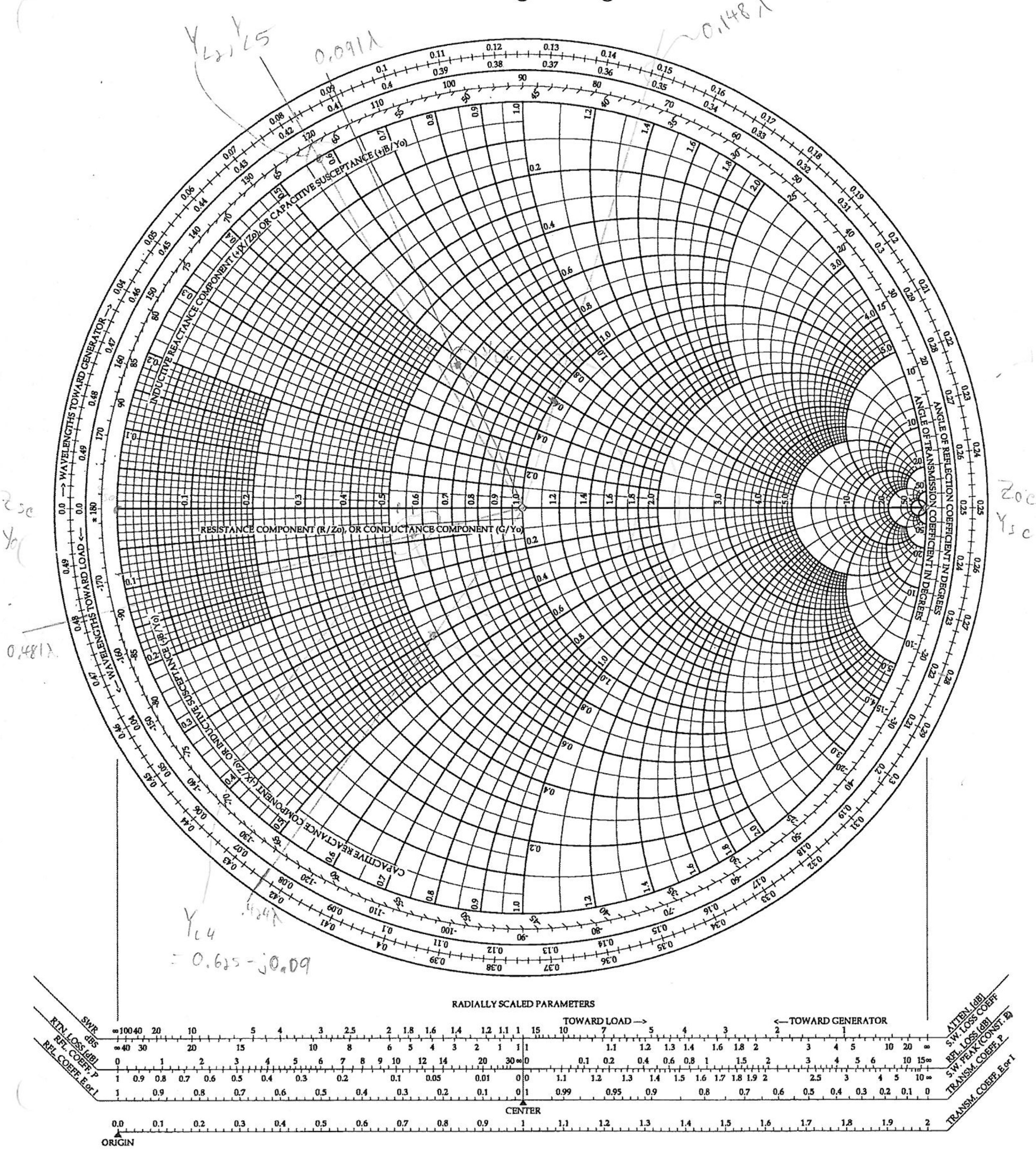
$\boxed{SWR = 2.225}$ $\boxed{z_{L7} = 332.4 + j201.5}$

12

Prob. 15.13

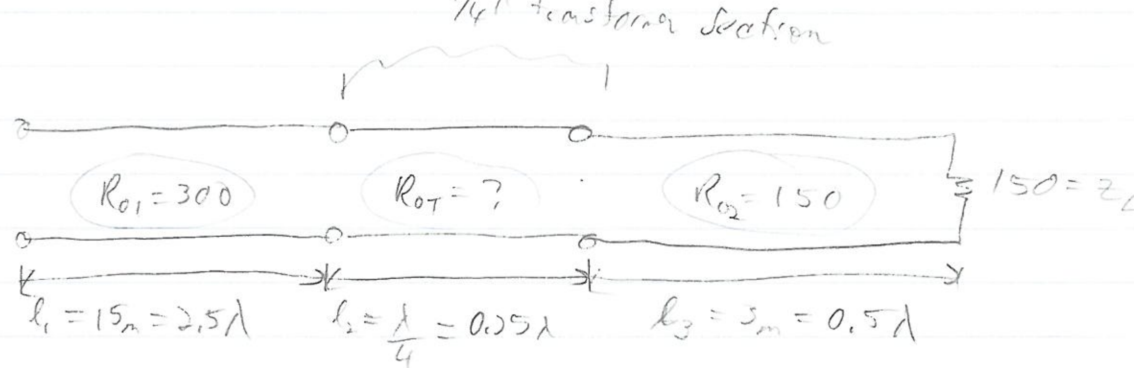
The Complete Smith Chart

Black Magic Design



13

15.15)



$f = 50\,MHz$, no dielectric $\Rightarrow u = 3E8\ m/s$

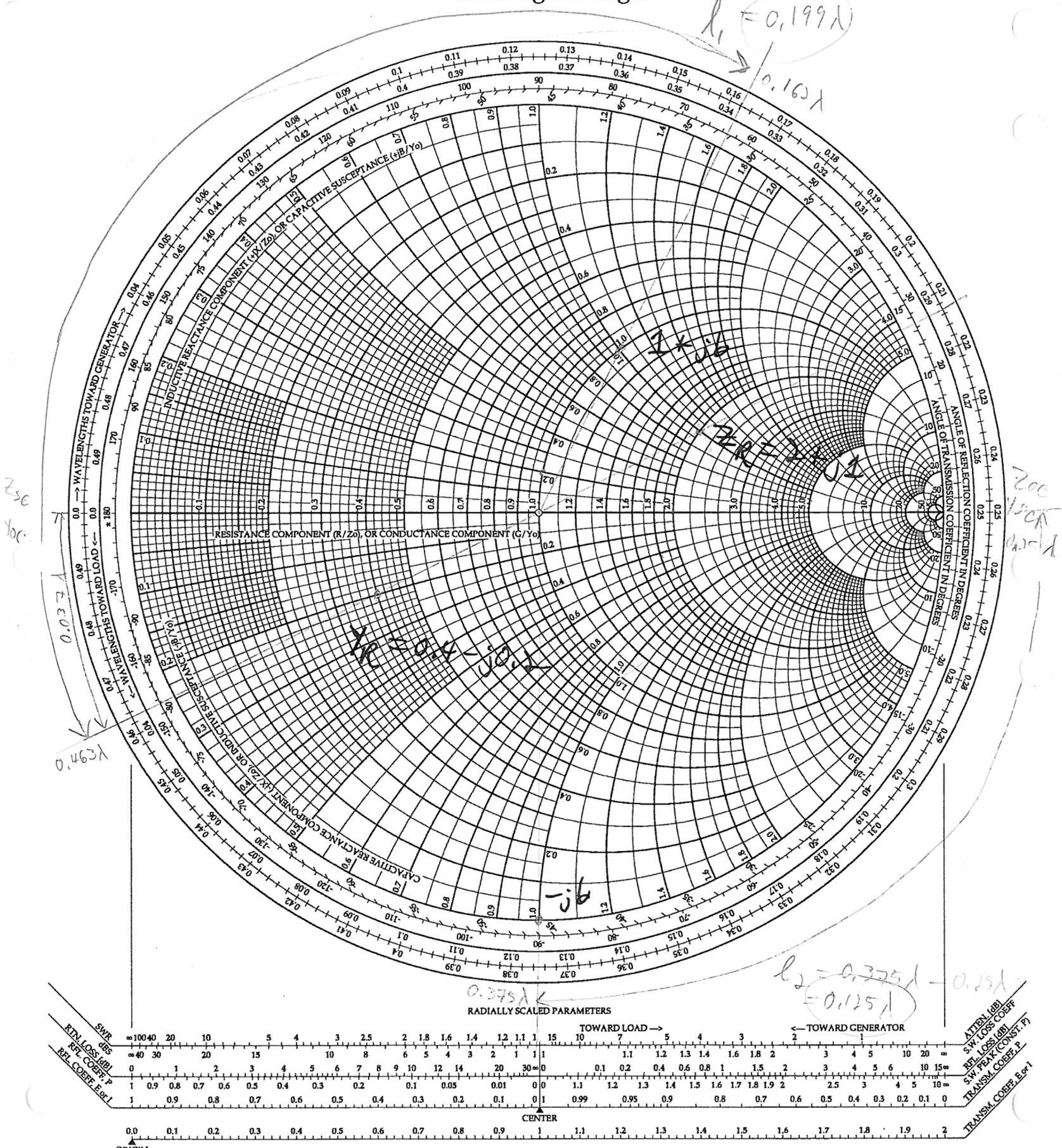
$\lambda = \frac{3E8}{50E6} = 6_m$

$R_{0T} = \sqrt{R_{01} R_{02}} = \sqrt{300(150)} = 212.1\,\Omega$

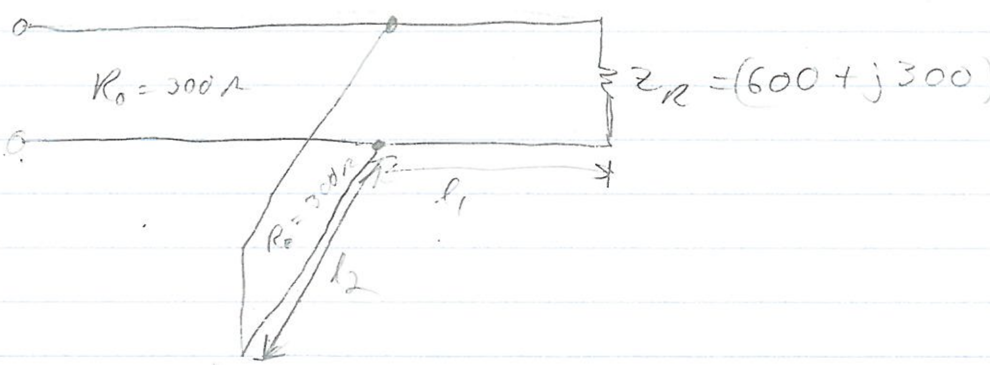
Prob. 5.17

The Complete Smith Chart

Black Magic Design



5.17)



$f = 600\ MHz$

$\lambda = \frac{3E8}{600E6} = 0.5\ m$

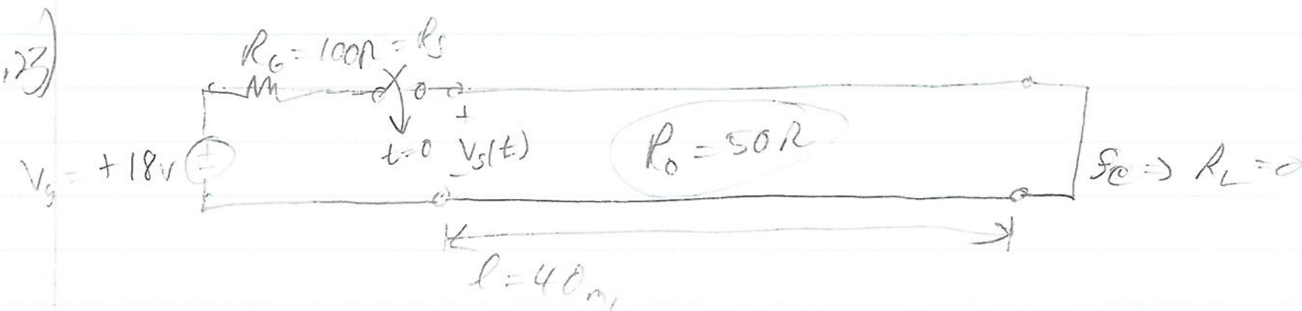
$z_R = \frac{(600 + j300)}{300} = 2 + j$

$l_1 = 0.199\lambda = 0.199(0.5) = \boxed{0.0995\ m}$

done

$l_2 = 0.125\lambda = 0.125(0.5) = \boxed{0.0625\ m}$

15.23)



$u = 2E8$ m/s

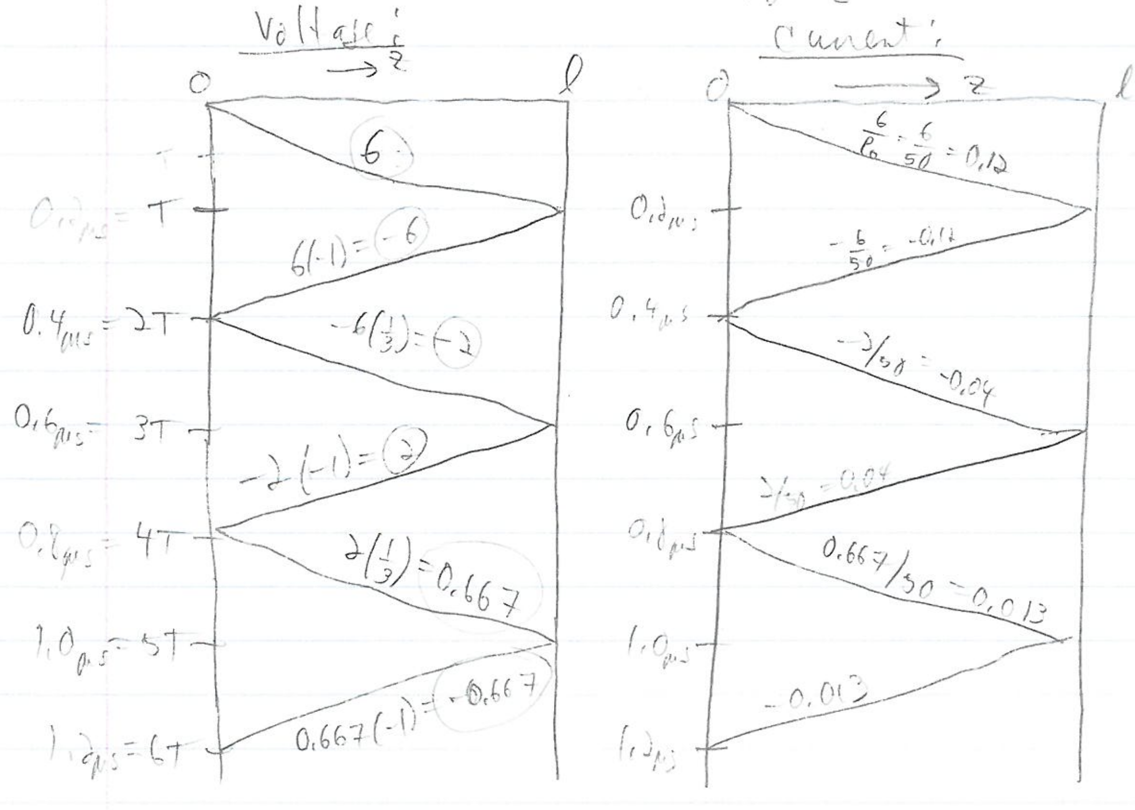
- sketch $V_S(t)$ from $t=0$ to $t = 2.5\mu S$

$$T = \frac{l}{u} = \frac{40m}{2E8\ m/s} = 0.2\mu S$$

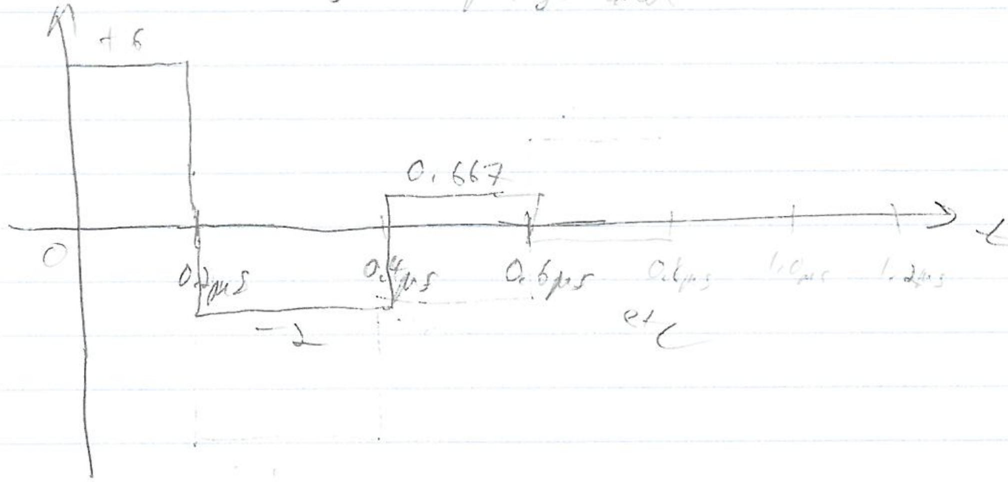
$$\Gamma_L = \frac{R_L - R_0}{R_L + R_0} = \frac{0-50}{0+50} = -1$$

$$\Gamma_S = \frac{R_S - R_0}{R_S + R_0} = \frac{100-50}{100+50} = \frac{50}{150} = \frac{1}{3}$$

$$V^+(z=0, t=0^+) = V_g(0^+)\frac{R_0}{R_0+R_S} = 18\left(\frac{50}{50+100}\right) = \frac{18}{3} = 6V$$



$V_s(t) \rightarrow$ located just after generator



Ch. 17: Antennas

17.1) A center-fed dipole antenna with a z-directed current has electrical length $\frac{L}{\lambda} \ll \frac{1}{30}$

a) Show that the current distribution may be assumed to be triangular in form: