Chapter 1: Vector Analysis

1.1) $\bar{M}(x_1, y_1, z_1)$, $\bar{N}(x_2, y_2, z_2)$

$$\overline{MN} = \underset{\text{from}}{\bar{N}} - \underset{\text{to}}{\bar{M}} = (x_2 - x_1)\hat{x} + (y_2 - y_1)\hat{y} + (z_2 - z_1)\hat{z}$$

1.3) in cylindrical coords, find the distance between $A = (5, \frac{3\pi}{2}, 0)$ and, $B = (5, \frac{\pi}{2}, 10)$

$(\rho, \phi, \hat{z})$

- convert to cartesian coords

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

- <u>for A</u>

$$\left.\begin{aligned} A_x &= 5\cos\left(\tfrac{3\pi}{2}\right) = 0 \\ A_y &= 5\sin\left(\tfrac{3\pi}{2}\right) = -5 \\ A_z &= 0 \end{aligned}\right\} A = -5\hat{y}$$

- <u>for B</u>

$$\left.\begin{aligned} B_x &= 5\cos\left(\tfrac{\pi}{2}\right) = 0 \\ B_y &= 5\sin\left(\tfrac{\pi}{2}\right) = 5 \\ B_z &= 10 \end{aligned}\right\} B = 5\hat{y} + 10\hat{z}$$

$$|B - A| = |(5+5)\hat{y} + (10-0)\hat{z}| = |10\hat{y} + 10\hat{z}|$$

$$= \sqrt{10^2 + 10^2} \doteq 14.14$$

1.5) $\overset{A_1}{\bar{A}} = 2\hat{x} + \overset{A_2}{4}\hat{y}, \quad \bar{B} = \overset{B_2}{6}\hat{y} - \overset{B_3}{4}\hat{z}$

- find the angle between $\bar{A}$ and $\bar{B}$

a) using the cross product

- use the identity: $|\bar{A} \times \bar{B}| = |\bar{A}||\bar{B}| \sin\theta \Rightarrow \theta = \sin^{-1}\dfrac{|\bar{A} \times \bar{B}|}{|\bar{A}||\bar{B}|}$

$$|\bar{A} \times \bar{B}| = \left| \hat{x}\begin{vmatrix} 4 & 0 \\ 6 & -4 \end{vmatrix} - \hat{y}\begin{vmatrix} 2 & 0 \\ 0 & -4 \end{vmatrix} + \hat{z}\begin{vmatrix} 2 & 4 \\ 0 & 6 \end{vmatrix} \right|$$

$$= \left| \hat{x}(-16-0) - \hat{y}(-8) + \hat{z}(12-0) \right|$$

$$= \left| -16\hat{x} + 8\hat{y} + 12\hat{z} \right| = \sqrt{(-16)^2 + 8^2 + 12^2}$$

$$|\bar{A} \times \bar{B}| = 21.54$$

$|\bar{A}| = \sqrt{2^2 + 4^2} = 4.47 = |\bar{A}|$, $|\bar{B}| = \sqrt{6^2 + (4)^2} = 7.21 = |\bar{B}|$

thus, $\theta = \sin^{-1}\left(\dfrac{21.54}{(4.47)(7.21)}\right) = 41.94° = \theta$

b) using the dot product

$$\cos\theta = \frac{\bar{A} \cdot \bar{B}}{|\bar{A}||\bar{B}|}$$

$$\bar{A} \cdot \bar{B} = \cancel{2}(0) + 4(6) + \cancel{0}(-4) = 24$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{24}{4.47(7.21)}\right) = 41.87°$$

1.7) Given: $\bar{A} = \hat{x} + \hat{y}$, $\bar{B} = \hat{x} + 2\hat{y}$, $\bar{C} = 2\hat{y} + \hat{z}$

– find $(\bar{A} \times \bar{B}) \times \bar{C}$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = \hat{x}\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - \hat{y}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + \hat{z}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 0\hat{x} - \hat{y}0 + \hat{z}(2-1) = \hat{z} = \bar{A} \times \bar{B}$$

$$(\bar{A} \times \bar{B}) \times \bar{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \hat{x}\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - \hat{y}\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + \hat{z}\begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix}$$

$$= \hat{x}(-2) - \hat{y}(0) + \hat{z}(0) = \boxed{-2\hat{x} = (\bar{A} \times \bar{B}) \times \bar{C}}$$

– find $\bar{A} \times (\bar{B} \times \bar{C})$

$$\bar{B} \times \bar{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{vmatrix} = \hat{x}\begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} - \hat{y}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \hat{z}\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$

$$= \hat{x}(2) - \hat{y}(1) + \hat{z}(2) = 2\hat{x} - \hat{y} + 2\hat{z} = \bar{B} \times \bar{C}$$

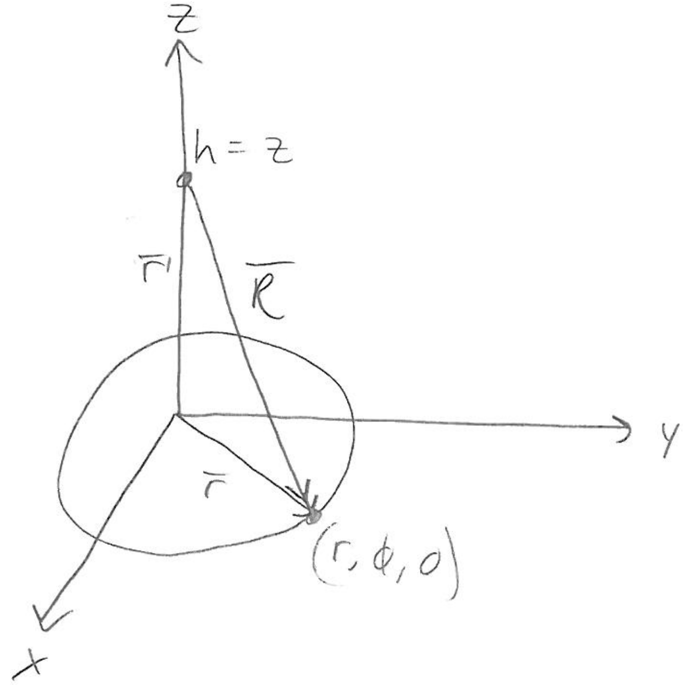
$$\bar{A} \times (\bar{B} \times \bar{C}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{vmatrix} = \hat{x}\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} - \hat{y}\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} + \hat{z}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= \hat{x}(2) - \hat{y}(2) + \hat{z}(-1-3)$$

$$= 2\hat{x} - 2\hat{y} - 4\hat{z}$$

– they are different is the order is different

1.9) Express the unit vector which points from $z=h$ on the z axis toward $(r,\phi,0)$ in cylindrical coordinates:



$\bar{r}' = 0\hat{\rho} + \phi\hat{\phi} + h\hat{z}$, $\bar{r} = r\hat{\rho} + \phi\hat{\phi}$

thus $\bar{r}-\bar{r}' = \underset{to}{\bar{r}} - \underset{from}{\bar{r}'} = (r\hat{\rho} + \phi\hat{\phi}) - (r\hat{\rho} + h\hat{z})$

$\bar{r}-\bar{r}' = r\hat{\rho} - h\hat{z}$

$x = \rho\cos\phi = r\cos 0 = r\hat{x}$

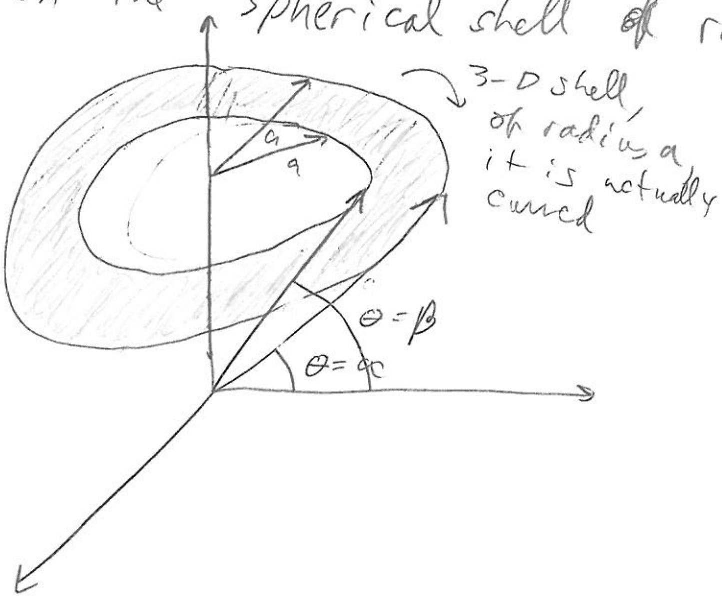
$y = \rho\sin\phi = r\sin 0 = 0\hat{y}$

$z = z = h = h\hat{z}$

and the unit vector $\bar{R}$

$$\bar{R} = \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|} = \boxed{\frac{r\hat{\rho} - h\hat{z}}{\sqrt{r^2+h^2}} = \bar{R}}$$

1.11) use the spherical coordinate system to find the area of the strip $\alpha \le \theta \le \beta$ on the spherical shell of radius a.



$\hat{r} + \hat{\theta} + \hat{\phi}$

- Since this is a shell area of a fixed radius a, we use the differential surface element

$$dS_r = r^2 \sin\theta \, d\theta \, d\phi$$

- so, the area is

$$A = \int_0^{2\pi} \int_{\alpha}^{\beta} a^2 \sin\theta \, d\theta \, d\phi = \int_0^{2\pi} \Big[a^2(-\cos\theta) \Big]_{\theta=\alpha}^{\beta} d\phi$$

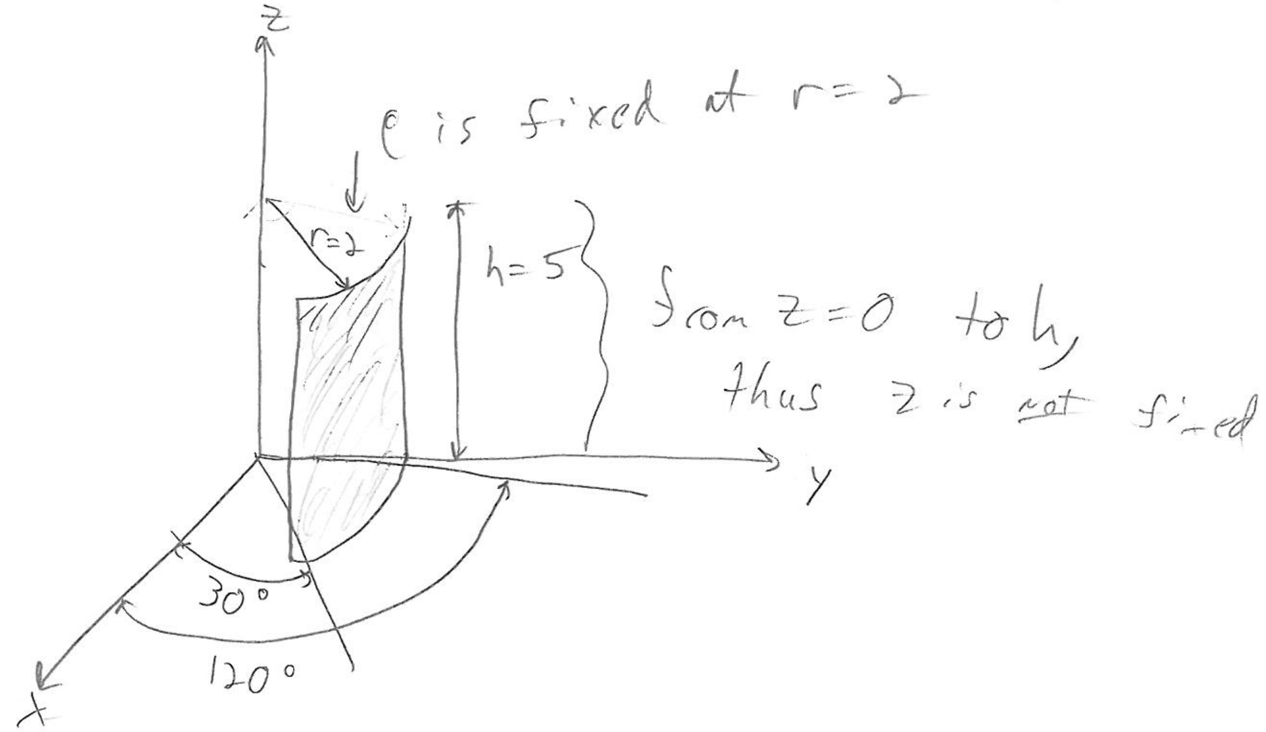
$$= \int_0^{2\pi} \left[a^2(-\cos\beta + \cos\alpha) \right] d\phi$$

$$= a^2 \int_0^{2\pi} (\cos\alpha - \cos\beta) \, d\phi = \boxed{2\pi a^2 (\cos\alpha - \cos\beta) = A}$$

- for $\alpha = 0$ and $\beta = \pi$

$$A = 2\pi a^2 (\cos 0 - \cos\pi) = \boxed{4\pi a^2 = A}$$

1.13) Use the cylindrical coord system to find the area of the curved surface of a right circular cylinder where $r = 2\,m$, $h = 5\,m$, $30° \le \phi \le 120°$



- Since $\hat{\rho}$ is fixed, the diff. surface element is

$$dS_\rho = \rho\, d\phi\, dz$$

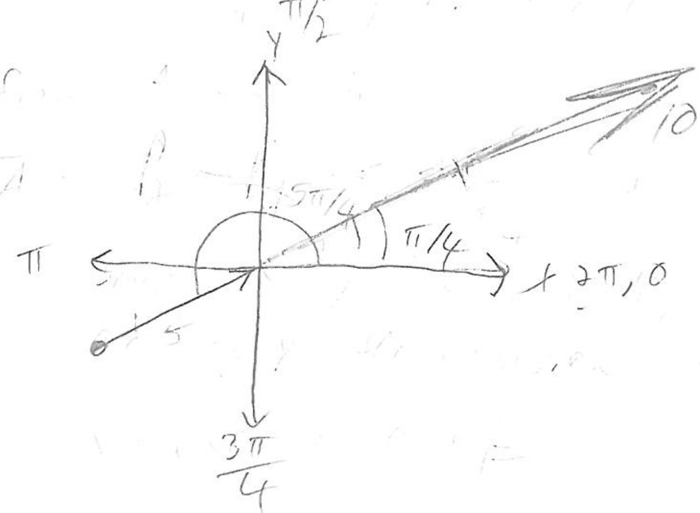
$$A = \int_0^5 \int_{30°}^{120°} \rho\, d\phi\, dz = \int_0^5 \int_{30°}^{120°} 2\, d\phi\, dz = 2\int_0^5 \int_{\pi/6}^{2\pi/3} d\phi\, dz$$

$$= 2(5)\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)$$

done

1.15) A vector of magnitude 10 points from $(5, 5\pi/4, 0)$ toward the origin. Express this vector in cartesian coords.

– plot the point



$$x = 10 \cos \frac{\pi}{4} = 5\sqrt{2}$$

$$y = 10 \sin \frac{\pi}{4} = 5\sqrt{2}$$

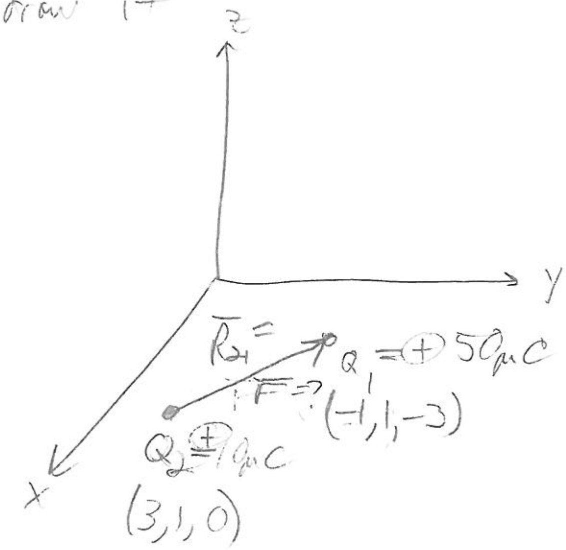
$$z = z = 0$$

$$\bar{A} = 5\sqrt{2}\,\hat{x} + 5\sqrt{2}\,\hat{y} + \emptyset\,\hat{z}$$

Chapter 2: Coulomb Forces and E field Intensity

2.1) Two point charges, $Q_1 = 50\mu C$ and $Q_2 = 10\mu C$ are located at $(-1,1,-3)$ m and $(3,1,0)$ m. Find the force on Q_1:

– draw it



$$\overline{R_{21}} = \underbrace{(-1,1,-3)}_{Q1} - \underbrace{(3,1,0)}_{Q2} = -4\hat{x} + 0\hat{y} - 3\hat{z}$$

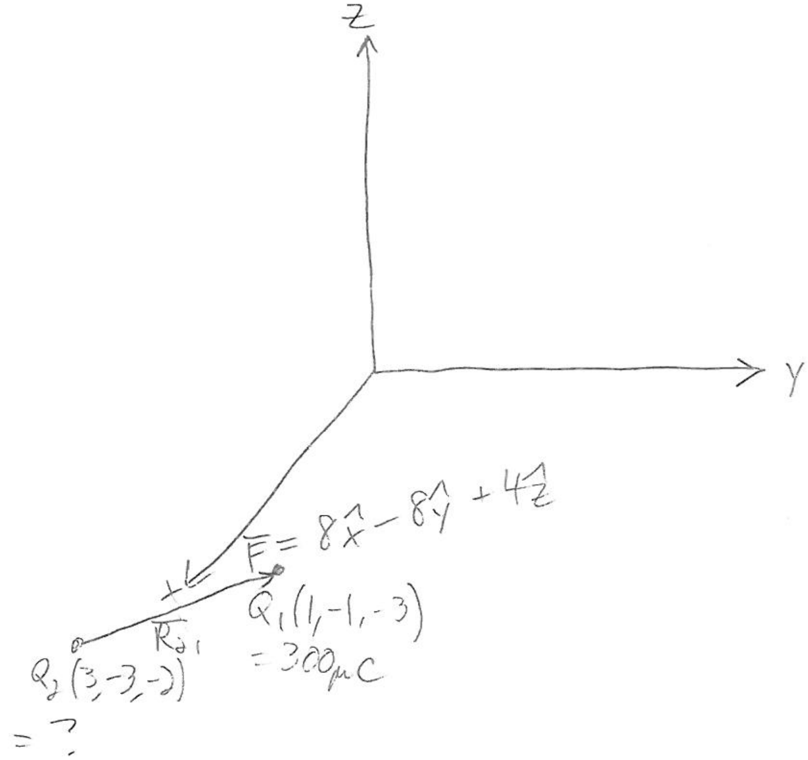
$$|\overline{R_{21}}| = \sqrt{(-4)^2 + (-3)^2} = 5$$

$$F_1 = \frac{Q_1 Q_2}{4\pi \varepsilon_0 |\overline{R_{21}}|^3} \overline{R_{21}} = \left[\frac{(50\mu)(10\mu)}{4\pi(8.854E-12)5^3}\right](-4\hat{x} - 3\hat{z})$$

$$= (0.03595)(-4\hat{x} - 3\hat{z}) = -0.1438\hat{x} - 0.1078\hat{y}$$

2.3) A point charge $Q_1 = 300\,\mu C$, located at $(1,-1,-3)\,m$, experiences a force $\bar{F}_1 = 8\hat{x} - 8\hat{y} + 4\hat{z}\ N$ due to a point charge Q_2 at $(3,-3,-2)\,m$.

Determine Q_2!



$$\bar{R}_{21} = (1,-1,-3) - (3,-3,-2) = -2\hat{x} + 2\hat{y} - \hat{z}$$

- apply Coulomb's law: $\quad |\bar{R}_{21}| = \sqrt{(-2)^2 + (2)^2 + (-1)^2} = 3$

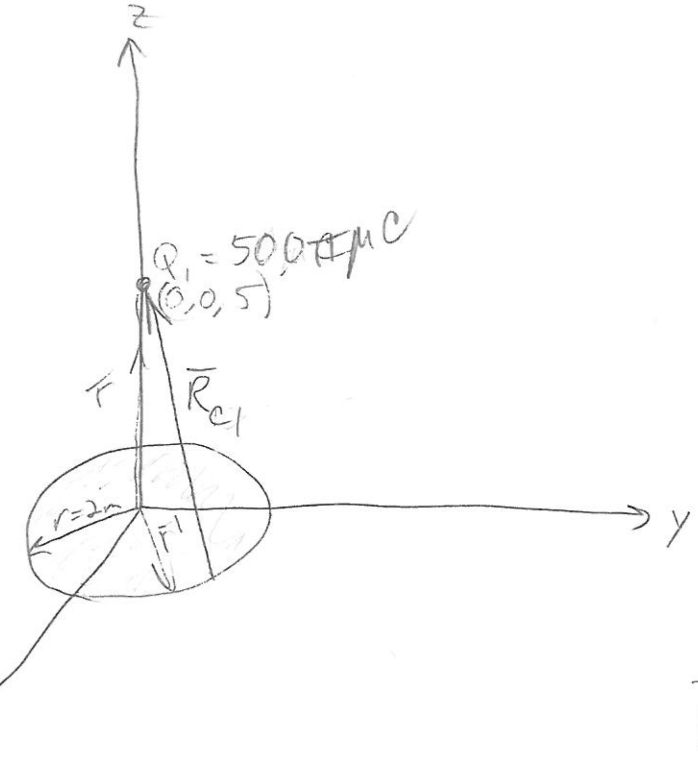
$$F_1 = \frac{Q_1 Q_2}{4\pi \varepsilon_0 |\bar{R}_{21}|^3} \bar{R}_{21}$$

$$8\hat{x} - 8\hat{y} + 4\hat{z} = \frac{(300\mu) Q_2}{4\pi (8.854E{-}12)\, 3^3} \left[-2\hat{x} + 2\hat{y} - \hat{z}\right]$$

$$8\hat{x} = -2\,Q_2\,(99.86E3)\,\hat{x} \Rightarrow Q_2 = -40\mu C$$

$$-8\hat{y} = 2\,Q_2\,(99.86E3)\,\hat{y} \Rightarrow Q_2 = -40\mu C$$

2.5) Find the force on a point charge of $500\pi\,\mu C$ at $(0,0,5)$m due to a charge of $500\pi\,\mu C$ that is uniformly distributed over circular disk $r=2$m, $z=0$m



$$\bar{R}_{c1} = \bar{r} - \bar{r}'$$

- Find the charge density

$$A = \pi r^2 = 4\pi$$

$$\rho_s = \frac{Q_1}{A} = \frac{500\pi\mu}{4\pi} = 125\,\mu C/m^2$$

- find the diff Force

$$dF = \frac{Q_1 \rho_s \, dS_\rho}{4\pi\varepsilon_0 |\bar{R}_{c1}|^3} \bar{R}_{c1}$$

where $dS_z = \rho\, d\rho\, d\phi$

- Find $\bar{R}_{c1}$

$$\bar{R}_{c1} = (0,0,5) - (\rho, \phi, \cancel{z})$$

$$= -\rho\hat{\rho} - \cancel{\phi\hat{\phi}} + 5\hat{z}$$

since we are on a disk of $r=2$, the sol is angle independent

→ Thus: $\bar{R}_{c1} = -\rho\hat{\rho} + 5\hat{z}$

$$|\bar{R}_{c1}| = \sqrt{\rho^2 + 25}$$

- Integrate:

$$F = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{2} \frac{(500\pi\mu)(\cancel{3.98\mu})^{125\mu}\, \rho\, d\rho\, d\phi}{4\pi(8.854E-12)(\sqrt{\rho^2+25})^3} \left[-\rho\hat{\rho} + 5\hat{z}\right]$$

3

the radial components of this integration will cancel because the integral over a disk area is $\emptyset$. Thus, we can simplify the integral in terms of the $\hat{z}$ comp.

$$\bar{F} = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{2} \frac{(500\pi\mu)\left(\cancel{3.98\mu}\ 125\mu\right)\rho\, d\rho\, d\phi}{4\pi(8.854E-12)(\rho^2+25)^{3/2}}\left(5\hat{z}\right)$$

$$\bar{F} = \cancel{280.9}\ 8823.7\,\hat{z}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{2}\frac{\rho\, d\rho\, d\phi}{(\rho^2+25)^{3/2}} = \cancel{280.9}\ 8823.7\,\hat{z}\int_{0}^{2}\frac{\rho\, d\rho}{(\rho^2+25)^{3/2}}$$

$$= \cancel{178.86\pi}\ 8823.7\,\hat{z}\left[\frac{-1}{\sqrt{\rho^2+25}}\right]_0^2 = \cancel{178.86\pi}\ 8823.7\left[\frac{1}{5}-\frac{\sqrt{29}}{29}\right]\hat{z}$$

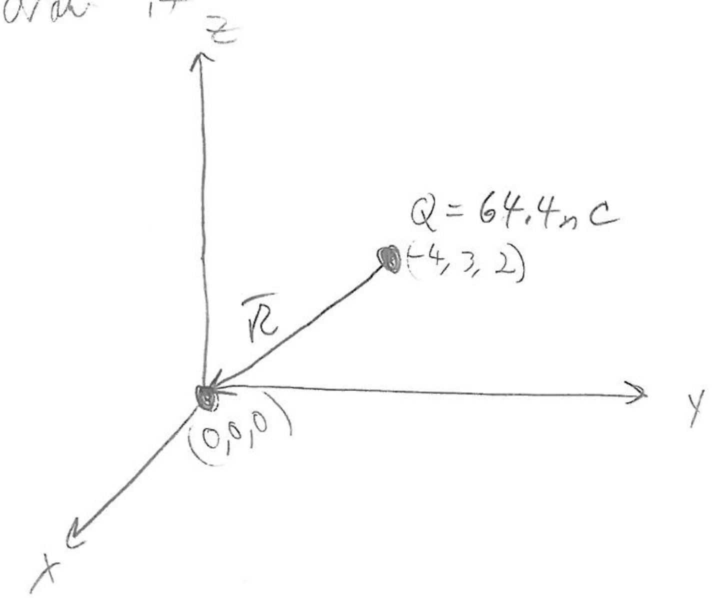
$$0.0143$$

$$\bar{F} = 126.2\,\hat{z}$$

4

2.7) Find E at the origin due to a point charge of 64.4 nC located at (-4, 3, 2) m in cartesian coordinates.

- draw it



$$\bar{R} = (0,0,0) - (-4,3,2) = 4\hat{x} - 3\hat{y} - 2\hat{z}$$

$$|\bar{R}| = \sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}$$

- use the formula for E feild of a pt. charge

$$E = \frac{Q}{4\pi\varepsilon_0 |\bar{R}|^2}\left(\frac{\bar{R}}{|\bar{R}|}\right) = \frac{(64.4n)}{4\pi(8.854E-12)(\sqrt{29})^2}\left(\frac{4\hat{x} - 3\hat{y} - 2\hat{z}}{\sqrt{29}}\right)$$

$$E = 14.83\hat{x} - 11.12\hat{y} - 7.41\hat{z}$$

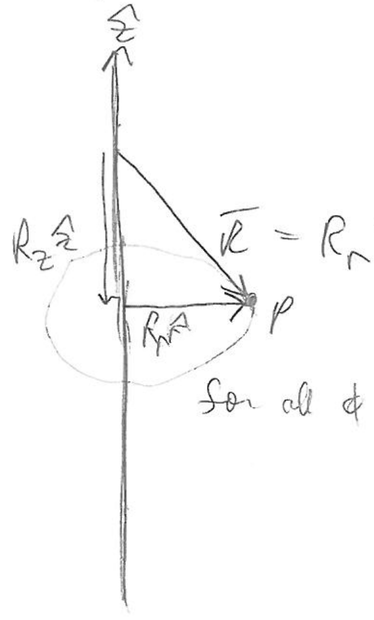
2.9) Charge is distributed uniformly along an infinite straight line with a constant density ρ_ℓ.

Develope the expression for E at the general pt. P.

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 |\bar{R}|^2}\left(\frac{\bar{R}}{|\bar{R}|}\right)$$

$$\Rightarrow \bar{E} = \int_{-\infty}^{\infty} \frac{\rho_s}{4\pi\varepsilon_0 |\bar{R}|^2}\left(\frac{\bar{R}}{|\bar{R}|}\right) d\ell$$

- define $\bar{R}$
  - cylindrical coords will be used, and we will place the infinite line charge on the $\hat{z}$ axis
  - draw it



$$\bar{R} = R_r\hat{r} + R_z\hat{z} \Rightarrow |\bar{R}| = \sqrt{R_r^2 + R_z^2}$$

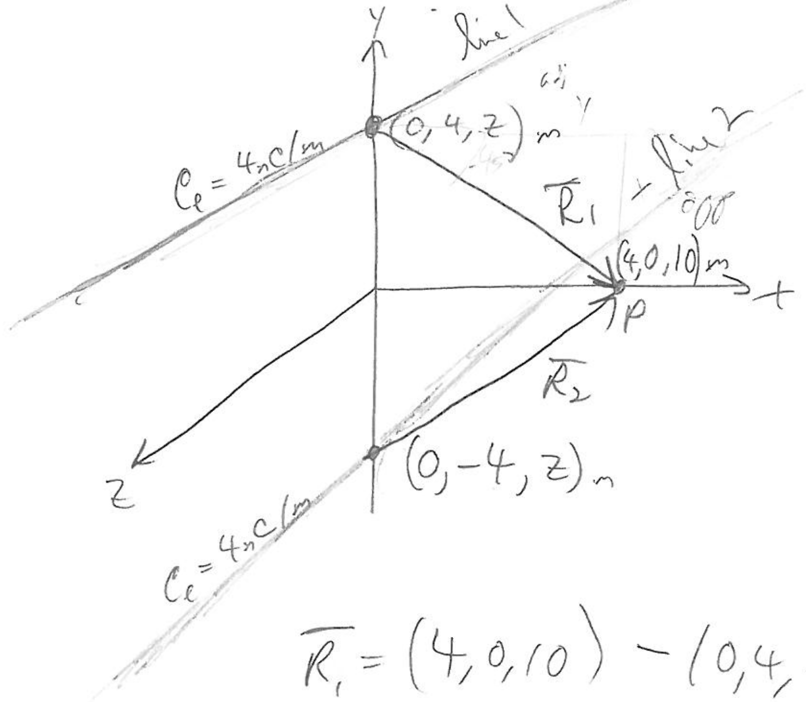
thus:

$$\bar{E} = \int_{-\infty}^{\infty} \frac{\rho_s}{4\pi\varepsilon_0 (R_r^2 + R_z^2)^{3/2}} (R_r\hat{r} + R_z\hat{z})\, d\ell$$

- because of symmetry, the $\hat{z}$ comp's cancel to $\emptyset$
  - explanation: for every dQ at z, there is an opposite and equal dQ at $-z$, thus the z comp's of the integral result cancel to $\emptyset$
- so

$$\boxed{\bar{E} = \int_{-\infty}^{\infty} \frac{\rho_s}{4\pi\varepsilon_0 \left(R_r^2 + R_z^2\right)^{3/2}} R_r \, dl \, \hat{r}}$$

2.11) Two uniform line charges of density $\rho_\ell = 4\,nC/m$ lie in the $x=0$ plane at $y = \pm 4\,m$. Find $\bar{E}$ at $(4,0,10)\,m$



$$\bar{R}_1 = (4,0,10) - (0,4,z) = 4\hat{x} - 4\hat{z}$$

$$\bar{R}_2 = (4,0,10) - (0,-4,z) = 4\hat{x} + 4\hat{z}$$

$$|\bar{R}_1| = \sqrt{4^2+4^2} = 4\sqrt{2}$$

$$|R_2| = \sqrt{4^2+4^2} = 4\sqrt{2}$$

- for one line charge in cylindrical coords

$$|\bar{E}| = \frac{\rho_\ell}{2\pi\varepsilon_0|R_m|} = \frac{4n}{2\pi(8.854E-12)[4\sqrt{2}]} = 12.71\ V/m$$

- thus, there is a 12.71 V/m contribution from each line charge. Since there are two of them, we must reduce each into cartesian vector form, and add to get full result (superposition)

- for line 1

- do a cylindrical to cartesian cood xfer

let $\rho = |\bar{E}| = 12.71$

$$x = \rho \cos\phi = 12.71 \cos(-45^\circ) = 8.98\hat{x}$$

$$y = \rho \sin\phi = 12.71 \sin(-45^\circ) = -8.98\hat{y}$$

$$z = z = 0$$

- thus, for line 1's E feild contribution:

$$\bar{E}_1 = 8.98\hat{x} - 8.98\hat{y} \ \mathrm{V/m}$$

- for line 2

let $\rho = |\bar{E}| = 12.71$

$$x = 12.71 \cos(+45^\circ) = 8.98\hat{x}$$

$$y = 12.71 \sin(+45^\circ) = 8.98\hat{y}$$

$$z = 0$$

$$\bar{E}_2 = 8.98\hat{x} + 8.98\hat{y}$$

- finally, the total superposition from both lines:

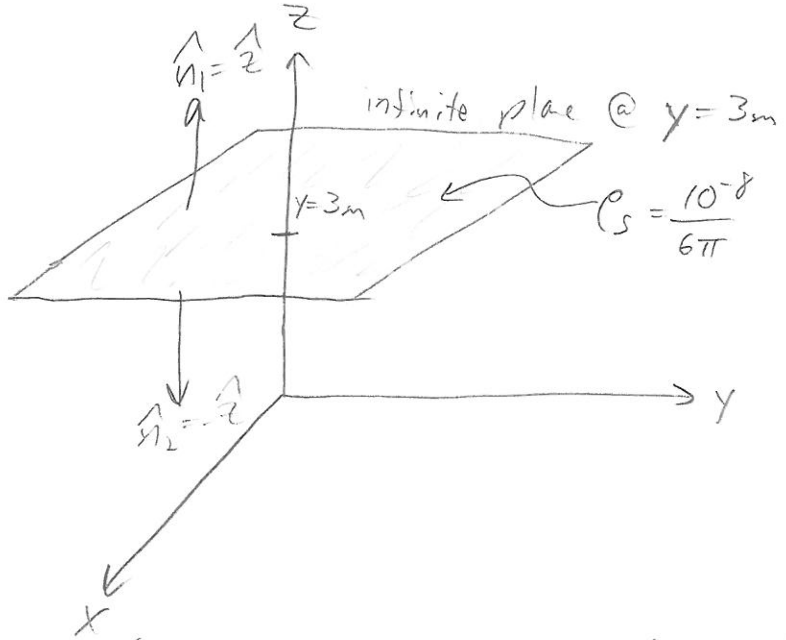
$$\bar{E}_T = \bar{E}_1 + \bar{E}_2 = (8.98\hat{x} - 8.98\hat{y}) + (8.98\hat{x} + 8.98\hat{y})$$

$$= \boxed{17.96\hat{x} = \bar{E}_T}$$

2.13) The plane $y = 3m$ contains a uniform charge distribution of density $\rho_s = \frac{10^{-8}}{6\pi}$ C/m^2.

Determine E at all points:

- draw it



- this is a stadad charge configuration, so use

$$E = \frac{\rho_s}{2\varepsilon_0}\hat{n}$$

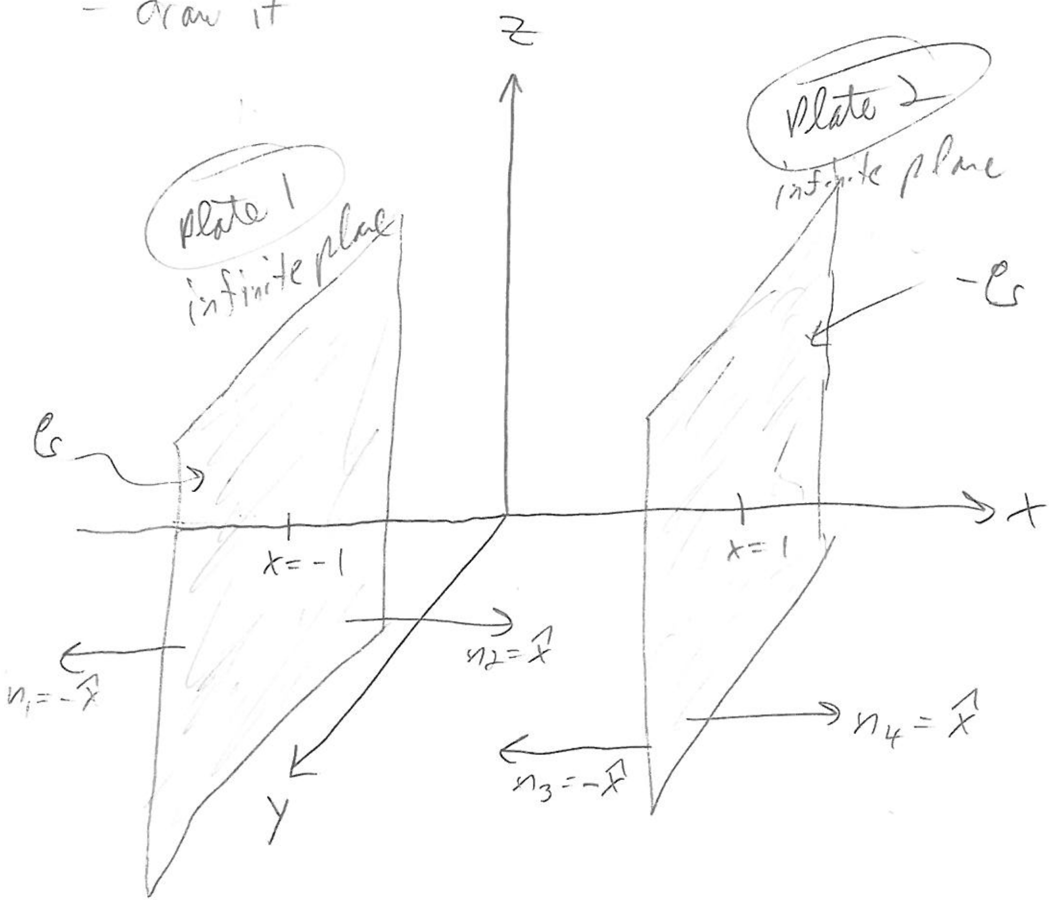
$\hat{n}_1 = \hat{z}$, $\hat{n}_2 = -\hat{z}$ for $z < 3m$

for $z > 3m$

$$E = \begin{cases} \dfrac{(10^{-8}/6\pi)}{2\varepsilon_0}\hat{z} & \text{for } y > 3m \\[2ex] \dfrac{(10^{-8}/6\pi)}{2\varepsilon_0}(-\hat{z}) & \text{for } y < 3m \end{cases}$$

2.15) an infinite sheet of charge with ρ_s is located at $x=-1$, another with charge $-\rho_s$ is located at $x=1$. Determine $\bar{E}$ in all regions:

- draw it



- this is a standard charge configuration, so

$$\bar{E} = \frac{\rho_s}{2\varepsilon_0}\hat{n}$$

- the total E feild must be found in each of the 3 regions

- note: Both E feilds due to plate 1 and plate 2 must be summed because these are not metal plates

- for $x < -1$

$$\bar{E} = \frac{\rho_s}{2\varepsilon_0}(-\hat{x}) + \frac{(-\rho_s)(-\hat{x})}{2\varepsilon_0} = \emptyset$$

- for $-1 < x < 1$

$$\bar{E} = \frac{\rho_s}{2\varepsilon_0}\hat{x} + \frac{(-\rho_s)}{2\varepsilon_0}(-\hat{x})$$

$$= \frac{2\rho_s}{2\varepsilon_0}\hat{x} = \frac{\rho_s}{\varepsilon_0}\hat{x} = \bar{E}$$

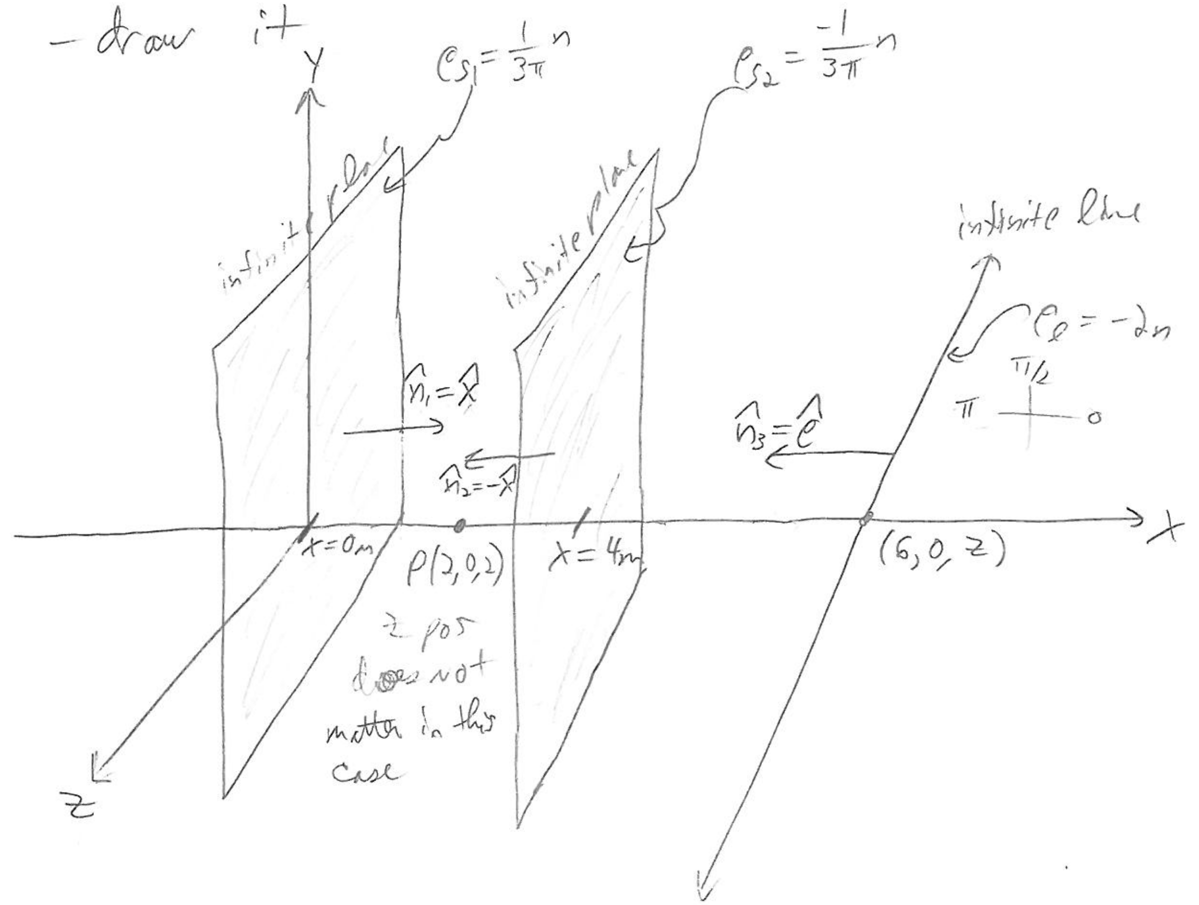
- for $x > 1$

$$\bar{E} = \frac{(-\rho_s)(\hat{x})}{2\varepsilon_0} + \frac{\rho_s(\hat{x})}{2\varepsilon_0} = \emptyset$$

2.17) Determine E at $(2,0,2)_m$ due to 3 standard charge distributions:

1. a uniform <u>sheet</u> at $x = \emptyset m$ with $\rho_{s1} = \frac{1}{3\pi}\ nC/m^2$
2. a uniform sheet at $x = 4m$ with $\rho_{s2} = \frac{-1}{3\pi}\ nC/m^2$
3. a uniform line at $\overset{x\ y\ z}{(6,0,z)}$ with $\rho_\ell = -2nC/m$

- draw it



- these are all standard charge configurations, thus the total E field is:

$$E = \frac{\rho_{s1}}{2\varepsilon_0}\hat{n}_1 + \frac{\rho_{s2}}{2\varepsilon_0}\hat{n}_2 + \frac{\rho_\ell}{2\pi\varepsilon_0 |R_\ell|}\hat{n}_3$$

$\hat{n}_3 = \hat{\rho} = \hat{x}\cos 180^\circ + \hat{y}\sin 180^\circ = -\hat{x}$

$R_\ell = |-4\hat{x}| = 4$

$$E = \frac{(1/3\pi\ E{-9})(\hat{x})}{2\varepsilon_0} + \frac{(-1/3\pi\ E{-9})(-\hat{x})}{2\varepsilon_0}$$

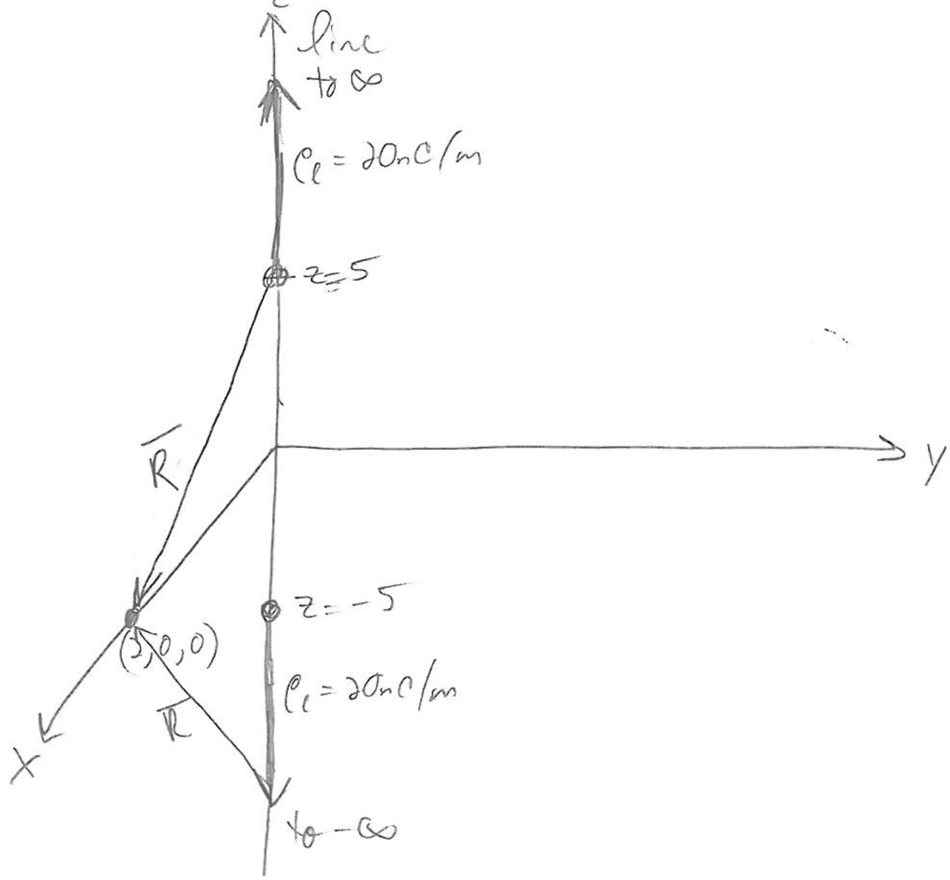
$$+ \frac{(-2E{-9})}{2\pi\varepsilon_0(4)}(-\hat{x})$$

$\bar{E} = 20.97\,\hat{x}$ V/m

dac

2.19) Charge is distributed along the z axis from $z = 5m$ to ∞, and from $z = -5m$ to $-\infty$ with a charge density of $\rho_\ell = 20 nC/m$. Find $\bar{E}$ at $(\overset{x}{2}, \overset{y}{0}, \overset{z}{0})$:

- draw it



$$\bar{R} = (2,0,0) - (0,0,z) = 2\hat{x} - z\hat{z} = \bar{R}$$ covers both lines

$$|\bar{R}| = \sqrt{2^2 + z^2} = \sqrt{4 + z^2}$$

- apply coulomb's:

$$d\bar{E} = \frac{(20n)}{4\pi\varepsilon_0(4+z^2)}\left(\frac{2\hat{x} - z\hat{z}}{\sqrt{4+z^2}}\right)$$

- because of symmetry, the $\hat{z}$ comp. cancels out of the integral

- because there are 2 lines, we have 2 different integral equations:

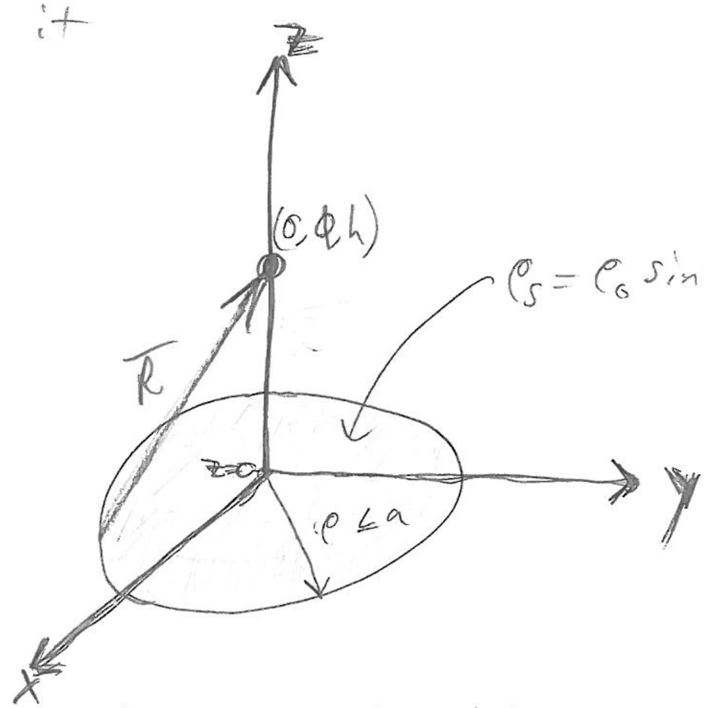
$$\bar{E} = \left[\int_{5}^{\infty} \frac{20n}{4\pi\varepsilon_0 (4+z^2)^{3/2}} dz + \int_{-\infty}^{-5} \frac{20n}{4\pi\varepsilon_0 (4+z^2)^{3/2}} dz \right] \hat{x}$$

top line   bott line

$$E = 179.75 \left[\int_{5}^{\infty} \frac{dz}{(4+z^2)^{3/2}} + \int_{-\infty}^{-5} \frac{dz}{(4+z^2)^{3/2}} \right] \hat{x}$$

2.21) a charge lies on the circular disk $r \leq a$, $z = 0$ with density $\rho_s = \rho_0 \sin^2\phi$. Determine E at $(\overset{\rho}{0}, \overset{\phi}{\phi}, \overset{z}{h})$

- draw it



- determine $\overline{R}$

$$\overline{R} = (0, \phi, h) - (\underset{\text{for } \rho \leq a}{\rho}, \phi, 0) = -\rho\hat{\rho} + h\hat{z}$$

$$|\overline{R}| = \sqrt{\rho^2 + h^2}$$

- apply Coulomb's law

$$dE = \frac{\rho_s}{4\pi\varepsilon_0 |\overline{R}|^2}\left(\frac{\overline{R}}{|\overline{R}|}\right)$$

$$dE = \frac{\rho_0 \sin^2\phi}{4\pi\varepsilon_0 (\rho^2 + h^2)^{3/2}}(-\rho\hat{\rho} + h\hat{z})$$

- although the charge distribution is not uniform, the $\hat{\rho}$ comp. is symmetrical, thus the $\hat{\rho}$ comp is $\emptyset$

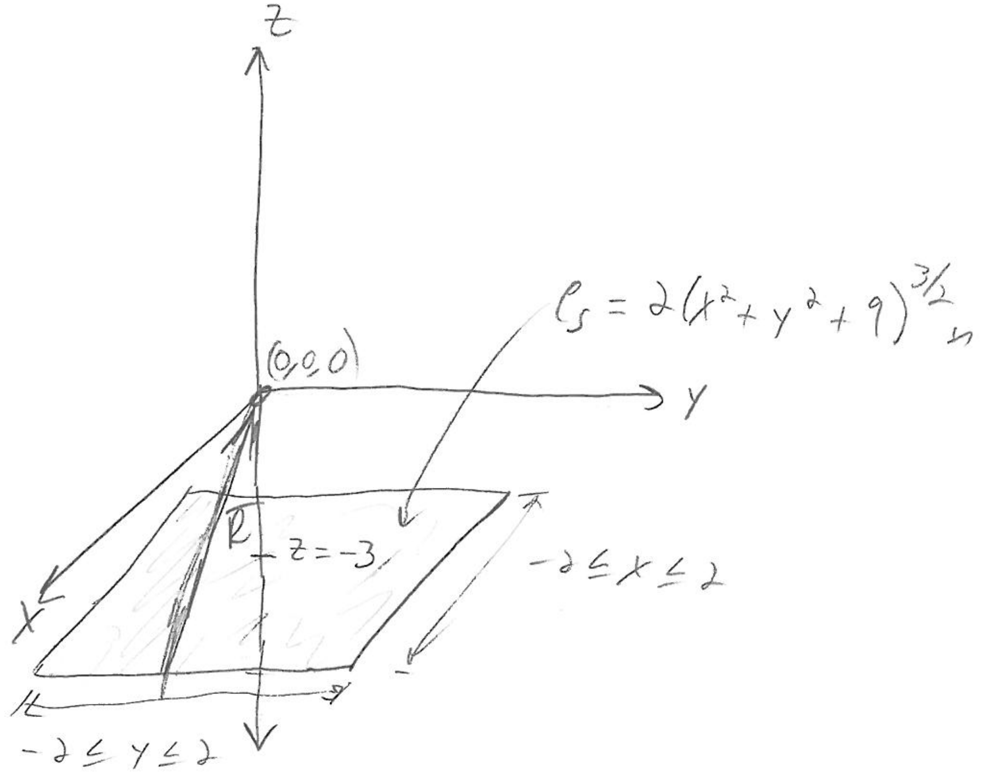
- thus the diff opp $dS_z = \rho \, d\rho \, d\phi$

$$\boxed{\bar{E} = \int_0^{2\pi} \int_0^{a} \frac{\rho_0 \sin^2\phi \, h}{4\pi \varepsilon_0 (\rho^2 + h^2)^{3/2}} \rho \, d\rho \, d\phi (\hat{z})}$$

2.23) Charge lies in the $z = -3$ m plane in the form of a square sheet defined by $-2 \leq x \leq 2$ m, $-2 \leq y \leq 2$ m with charge density $\rho_s = 2(x^2+y^2+9)^{3/2}$ nC/m².

Find $\bar{E}$ at the origin:

- draw it



$$\bar{R} = (0,0,0) - (x,y,-3) = -x\hat{x} - y\hat{y} + 3\hat{z}$$

$$|\bar{R}| = \sqrt{x^2+y^2+9}$$

$$d\bar{E} = \frac{\rho_s}{4\pi\varepsilon_0 |\bar{R}|^2}\left(\frac{\bar{R}}{|\bar{R}|}\right) \longleftarrow \text{apply coulomb's}$$

$$d\bar{E} = \frac{2\cancel{(2x^2+y^2+9)^{3/2}}}{4\pi\varepsilon_0 \cancel{(x^2+y^2+9)^{3/2}}}(-x\hat{x} - y\hat{y} + 3\hat{z})$$

$$d\bar{E} = \frac{1}{2\pi\varepsilon_0}\left(-x\hat{x} - y\hat{y} + 3\hat{z}\right)$$

- since x is from -2 to 2, it is symmetric and thus the $\hat{x}$ comp drops out
- since y is from -2 to 2, it is symmetric and thus the $\hat{y}$ comp drops out
- we are left only with the $\bar{E}$ feild in the $\hat{z}$ direction;

$$\boxed{\bar{E} = \int_{y=-2}^{2}\int_{x=-2}^{2} \frac{B_{E\cdot}9}{2\pi\varepsilon_0}\,dx\,dy\,(\hat{z})}$$

Chapter 3: Electric Flux and Gauss's Law

3.1) Find the charge in the volume defined by

$0 \le x \le 1m$, $0 \le y \le 1m$, and $0 \le z \le 1m$

if $\rho = 30x^2y \, (\mu C/m^3)$.

What charge occurs for the limits $-1 \le y \le 0m$?

– use the net charge in a region formula:

$$Q = \int_V \rho \, dv = \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1}(30E-6)x^2y \, dx\,dy\,dz$$

$$= (30E-6)\int_0^1\int_0^1\left[\frac{x^3}{3}y\right]_0^1 dy\,dz = (30E-6)\int_0^1\int_0^1 \frac{y}{3}\,dy\,dz$$

$$= (10E-6)\int_0^1\left[\frac{y^2}{2}\right]_0^1 dz = [5E-6]\left[x\right]_0^1 = 5\mu C$$

– for the change in volume limits for y,

$-1 \le y \le 0m$:

$$Q = \int_{z=0}^{1}\int_{y=-1}^{0}\int_{x=0}^{1}(30E-6)x^2y \, dx\,dy\,dz = (30E-6)\int_0^1\int_{-1}^0\left[\frac{x^3}{3}y\right]_0^1 dy\,dz$$

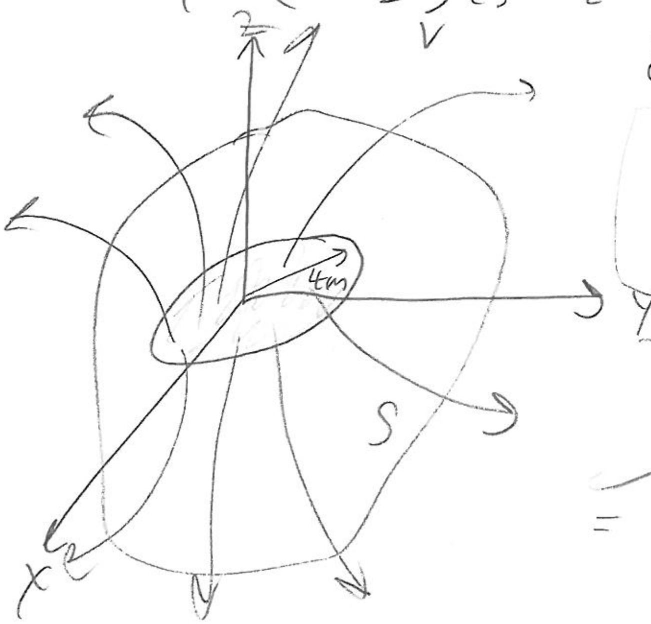
$$= (10E-6)\int_0^1\int_{-1}^0 y\,dy\,dz = (-5E-6)\int_0^1 dz = -5E-6 \; C$$

3.3) What net flux crosses the closed surface S, which contains a charge distribution in the form of a plane disk of radius 4 m with a density

$\rho_s = \frac{\sin^2\phi}{2\rho}$ (C/m^2)?

- by deff, net electric flux is:

$$\Psi = Q = \iint_V \rho_s \, ds_z = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{4} \frac{\sin^2\phi}{2\rho}\,\rho\, d\rho\, d\phi$$



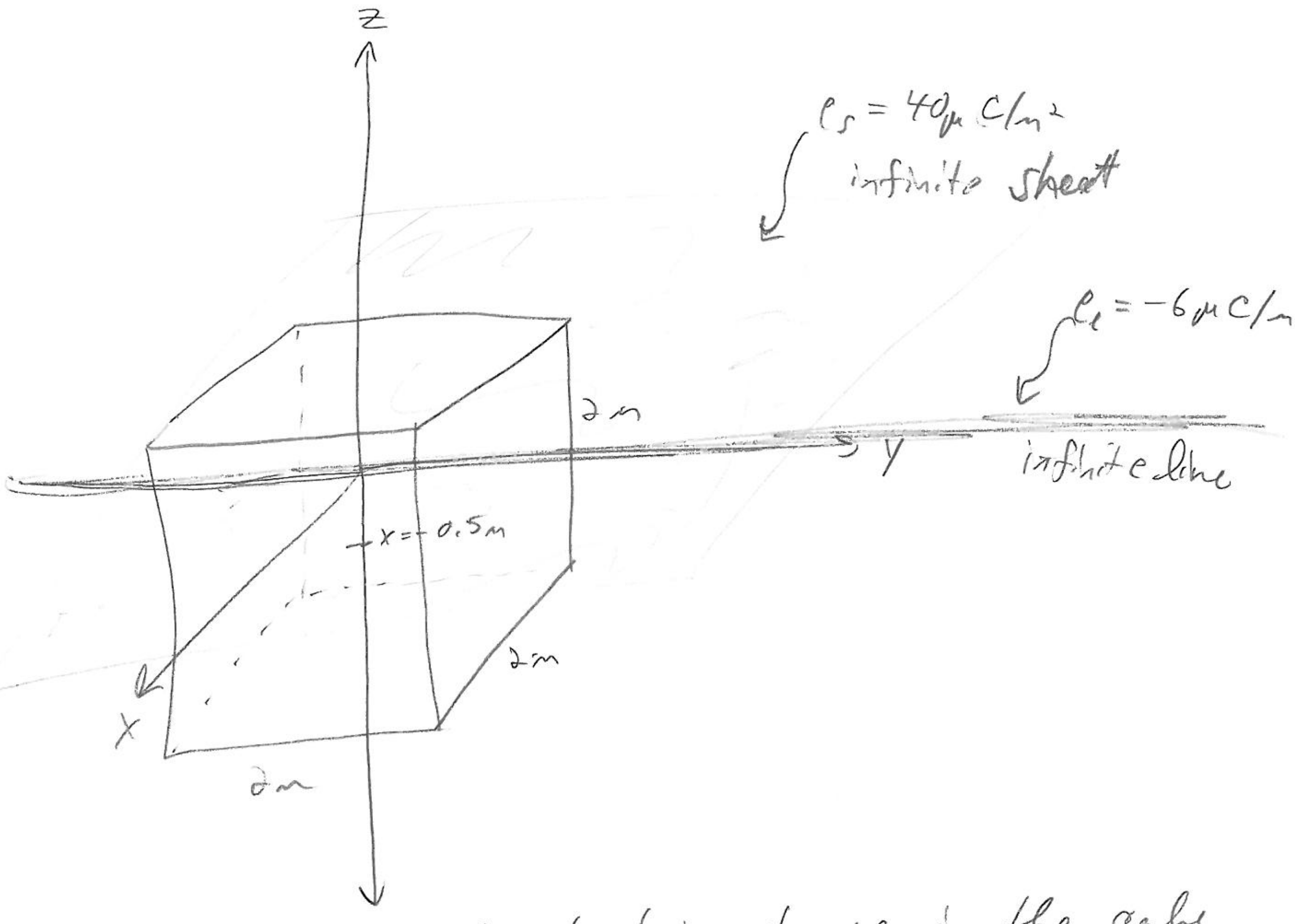
$$= \frac{1}{2}\int_0^{2\pi}\int_0^{4} \sin^2\phi \, \rho^{-1}\, d\rho\, d\phi \quad C$$

$$\cancel{= \frac{1}{2}\int_0^{2\pi} \sin^2\phi \left[\ln\rho\right]_0^4 d\phi}$$

$=$

$=$

3.5) Charge in the form of a plane sheet with density $\rho_s = 40\,\mu C/m^2$ is located at $z = -0.5\,m$. A uniform line charge of $\rho_\ell = -6\,\mu C/m$ lies along the y axis. What net flux crosses the surface of a cube 2m on an edge, centered at the origin:



- first the net electric charge in the cube region must be found:
  - net charge due to the sheet inside of the cube is

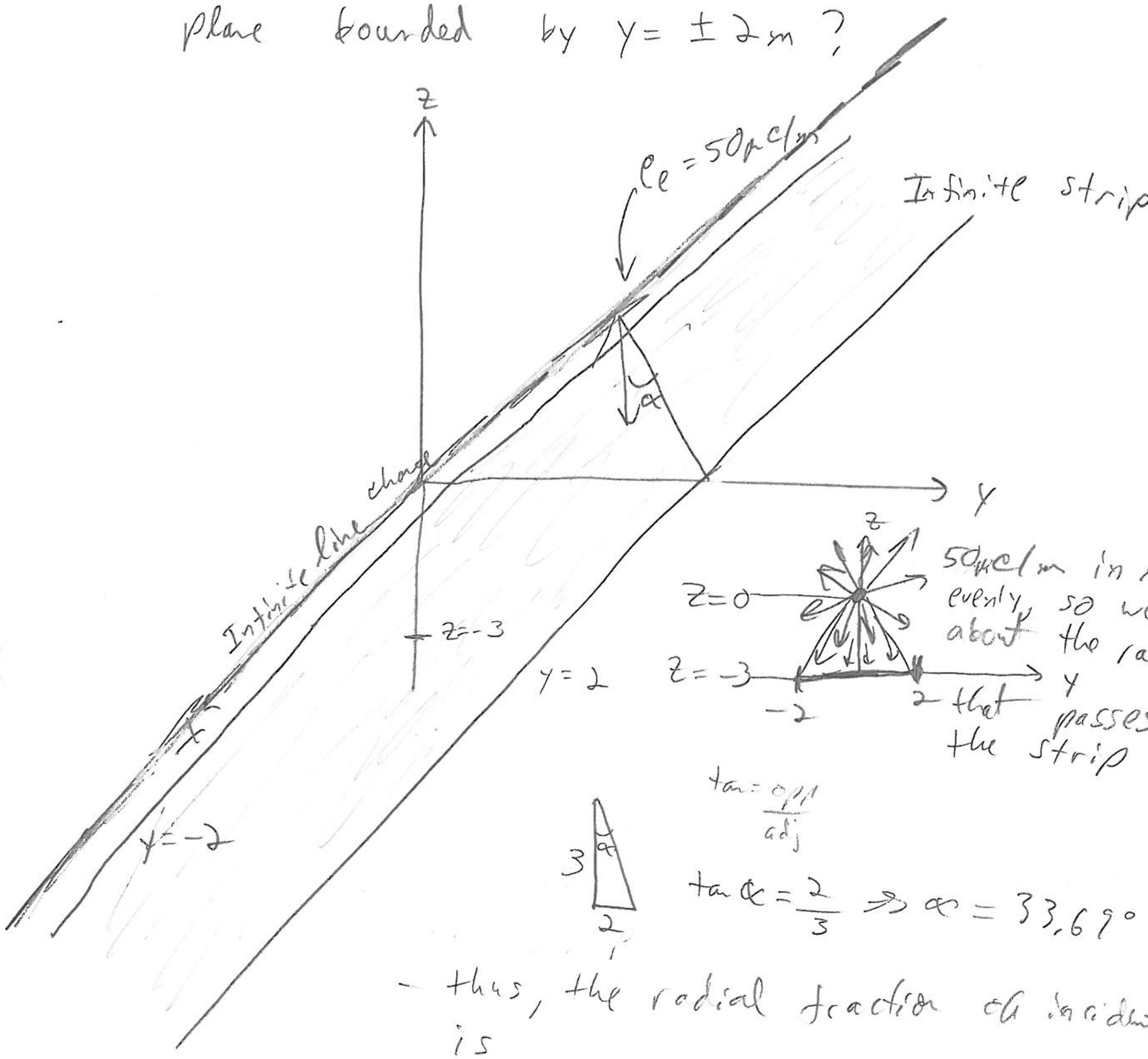
$$Q_1 = \int_V \rho_v\, dv = \int_{-1}^{1}\int_{-1}^{1} 40\mu\, dx\, dy = 4(40\mu) = 160\,\mu C$$

  - net charge due to the line inside of the cube is

$$Q_2 = \int_\ell \rho_\ell\, d\ell = \int_{-1}^{1} (-6\mu)\, dy = (-6\mu)[y]_{-1}^{1} = -12\,\mu C$$

- the entire net charge is the sum of the 2 found above $Q_T = Q_1 + Q_2 = 148\,\mu C = \Phi = \Psi$

3.7) A uniform line charge with $\rho_\ell = 50\,\mu C/m$ lies along the x axis. What flux per unit length, Ψ/L, crosses the portion of the $z=-3m$ plane bounded by $y=\pm 2m$?



$50\mu C/m$ in all directions evenly, so we only care about the radial fraction that passes thru the strip

$\tan = \frac{opp}{adj}$

$\tan\alpha = \frac{2}{3} \Rightarrow \alpha = 33.69°$

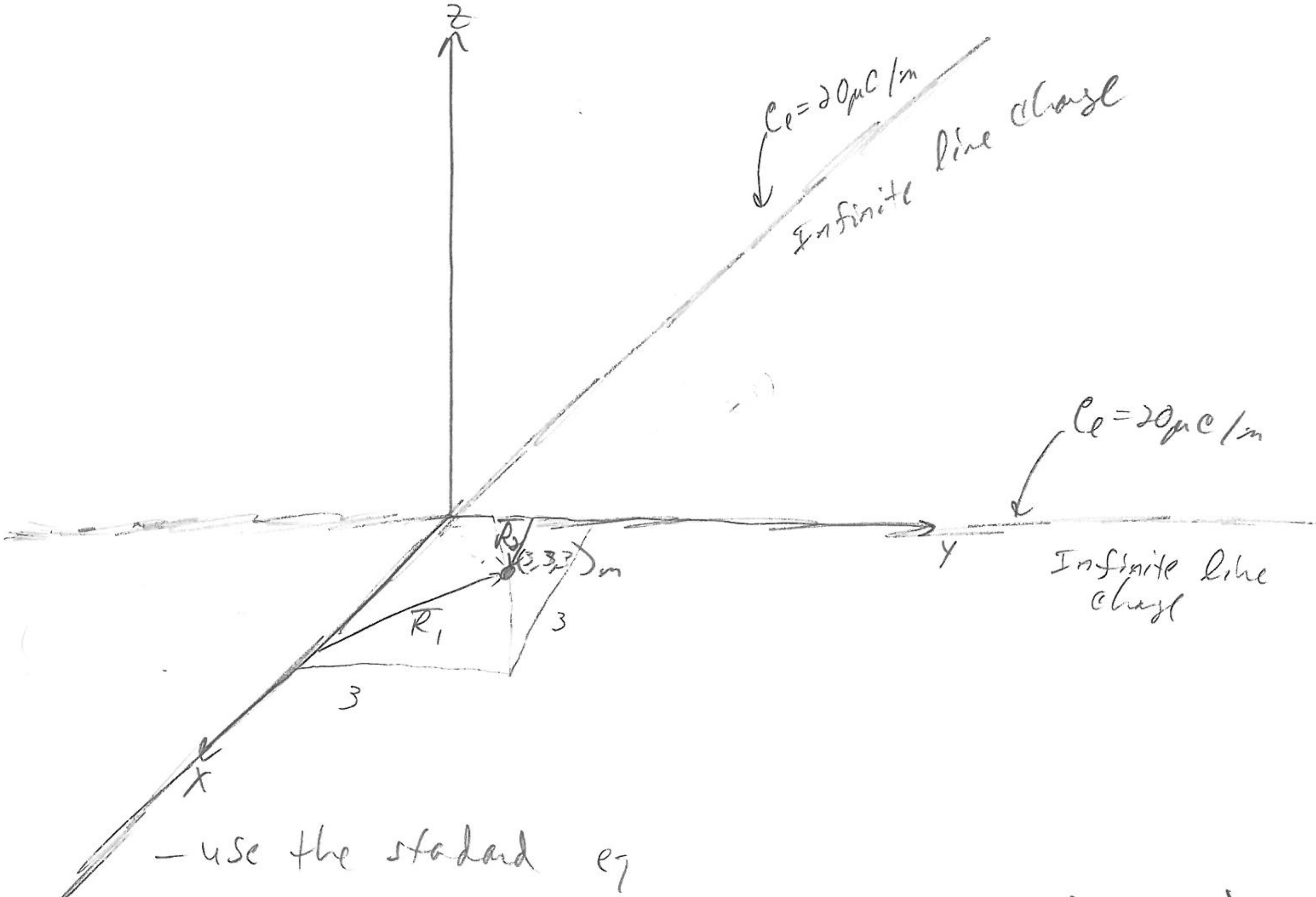
- thus, the radial fraction of incident charge is

$$\frac{2\alpha}{360} = \frac{2(33.69°)}{360} = 0.187$$

- and, finally the incident flux/m

$$\Psi/L = (0.187)(50\mu C) = 9.36\,\mu C/m$$

3.9) Two identical uniform line charges lie along the x and y axes with charge densities $\rho_\ell = 20\mu C/m$. Obtain $\bar{D}$ at $(3,3,3)$ m



- use the standard eq

since $\bar{D} = \varepsilon \bar{E} \Rightarrow \bar{D} = \varepsilon_0 E = \varepsilon_0 \left(\dfrac{\rho_\ell}{2\pi \varepsilon_0 |\bar{R}_\ell|} \right) \hat{\ell}$

$$\bar{D} = \frac{\rho_\ell}{2\pi |\bar{R}_\ell|} \hat{\ell}$$

- for $\bar{R}_1$, $\bar{R}_1 = (3,3,3) - (x,0,0) = \cancel{3}\hat{x} + 3\hat{y} + 3\hat{z}$ throw out x comp, since line charge is on x axis

$= 3\hat{y} + 3\hat{z} = \bar{R}_1$

$|R_\ell| = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{3^2 + 3^2}$

$|R_\ell| = 3\sqrt{2}$

- thus the contribution due to the x axis line charge is

$$\bar{D}_1 = \frac{20\mu}{2\pi(3\sqrt{2})} \left(\frac{3\hat{y} + 3\hat{z}}{3\sqrt{2}} \right) = (0.53E-6)(\hat{y} + \hat{z}) = \bar{D}_1$$

- and the contribution due to the y axis line charge is:

- find $\overline{R}_2$

$$\overline{R}_2 = (3,3,3) - (0,y,0) = 3\hat{x} - \cancel{y}\hat{y} + 3\hat{z}$$

throw out y comp since line charge is on y axis

$$\Rightarrow \overline{R}_2 = 3\hat{x} + 3\hat{z}$$

$$|\overline{R}_2| = \sqrt{3^2 + 3^2} = 3\sqrt{2} = |R_e|$$

- thus

$$\overline{D}_2 = \frac{20\mu}{2\pi \, 3\sqrt{2}} \left(\frac{\cancel{3}\hat{x} + \cancel{3}\hat{z}}{\cancel{3}\sqrt{2}} \right) = \frac{20\mu}{4\pi} (\hat{x} + \hat{z})$$

$$\overline{D}_2 = 0.53E{-}6 \, (\hat{x} + \hat{z})$$

- the total flux density is the vector sum:

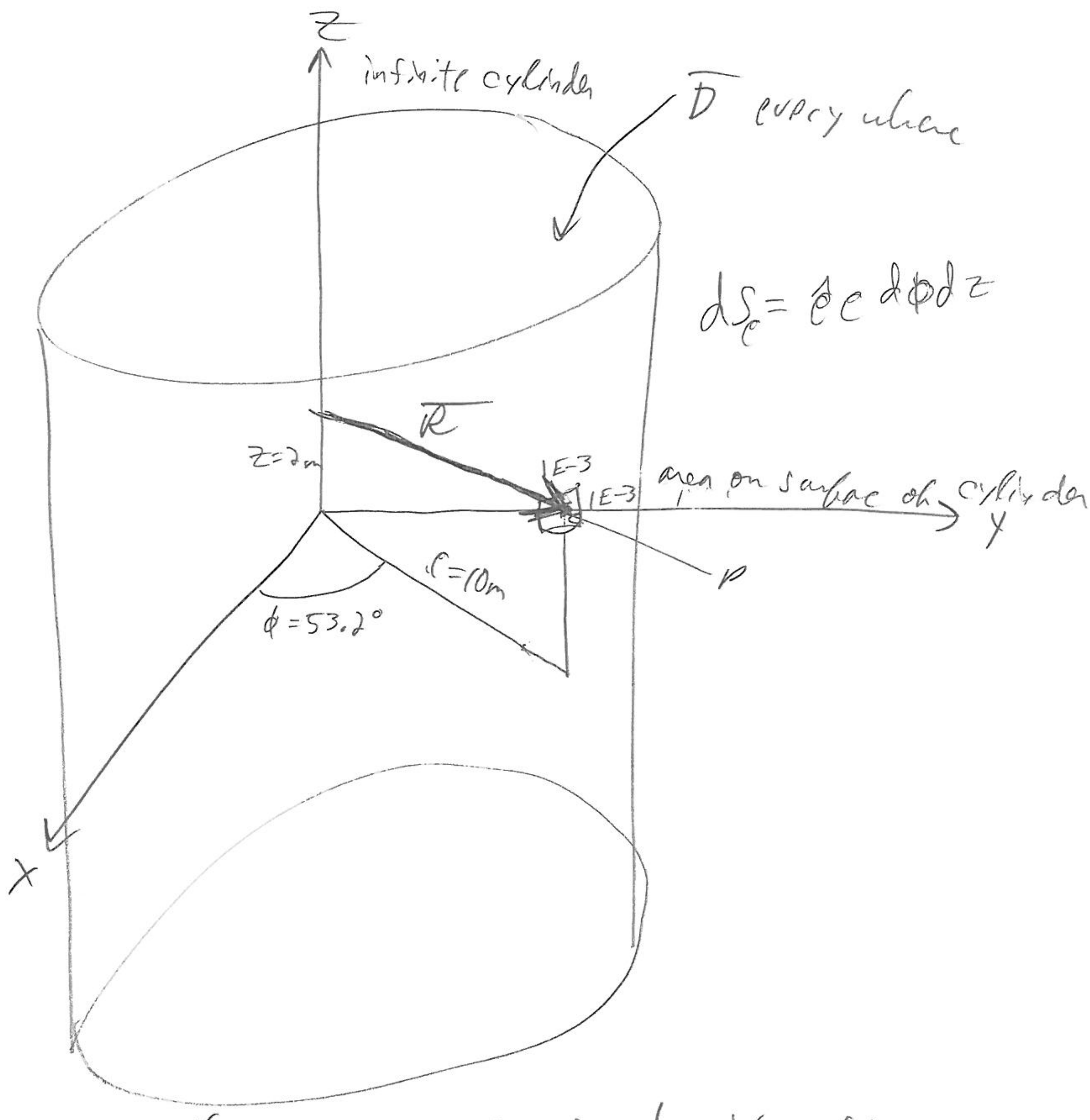
$$\overline{D}_T = \overline{D}_1 + \overline{D}_2 = 0.53E{-}6 \left[(\hat{y} + \hat{z}) + (\hat{x} + \hat{z}) \right]$$

$$\boxed{\overline{D}_T = 0.53E{-}6 \, (\hat{x} + \hat{y} + 2\hat{z})} \quad (\mu C/m^2)$$

3.11) Determine the flux crossing a 1mm by 1mm area on the surface of a cylindrical shell at $\rho = 10m$, $z = 2m$, $\phi = 53.2°$ if:

$$\bar{D} = 2x\hat{x} + 2(1-y)\hat{y} + 4z\hat{z} \quad (C/m^2)$$

- Since $\rho = 2m$, the shell is an infinite cylinder with a radius of 2m



- use this equation to find the flux

$$\text{flux} = d\Psi = \bar{D} \cdot d\bar{S} \quad (1)$$

$$\bar{D} = 2x\hat{x} + 2(1-y)\hat{y} + 4z\hat{z}$$

$\bar{D}$ at point $(10, 53.2°, 2)$:

$$x = \rho\cos\phi = 10\cos 53.2° = 6$$
$$y = \rho\sin\phi = 10\sin 53.2° = 8$$
$$z = z = 2$$

thus $P = \overset{x\ y\ z}{(6, 8, 2)}$ in cartesian coords, and $\bar{D}$ at P is

$$\bar{D}(P) = 12\hat{x} - 14\hat{y} + 8\hat{z}$$

- since the cylinder is infinite on the z axis we can ignore the $\hat{z}$ comp when determining $\bar{R}$

$$\bar{R} = \bar{P} - 0 = (6, 8, 2) - (0, 0, 2) = 6\hat{x} + 8\hat{y} = \bar{R}$$

$$|\bar{R}| = \sqrt{6^2 + 8^2} = 10 = |\bar{R}|$$

- in finding $d\bar{S}$, we can ignore the outside curvature of the cylinder because the cylinder has such a large radius of 10m and the area we are dealing with is very small, $1\,mm^2$

$$A = (1E-3)(1E-3) = (1E-6)$$

$$d\bar{S} = A\left(\frac{\bar{R}}{|\bar{R}|}\right) = (1E-6)\left(\frac{6\hat{x} + 8\hat{y}}{10}\right) = (1E-6)(0.6\hat{x} + 0.8\hat{y}) = d\bar{S}$$

- Apply ① from above:

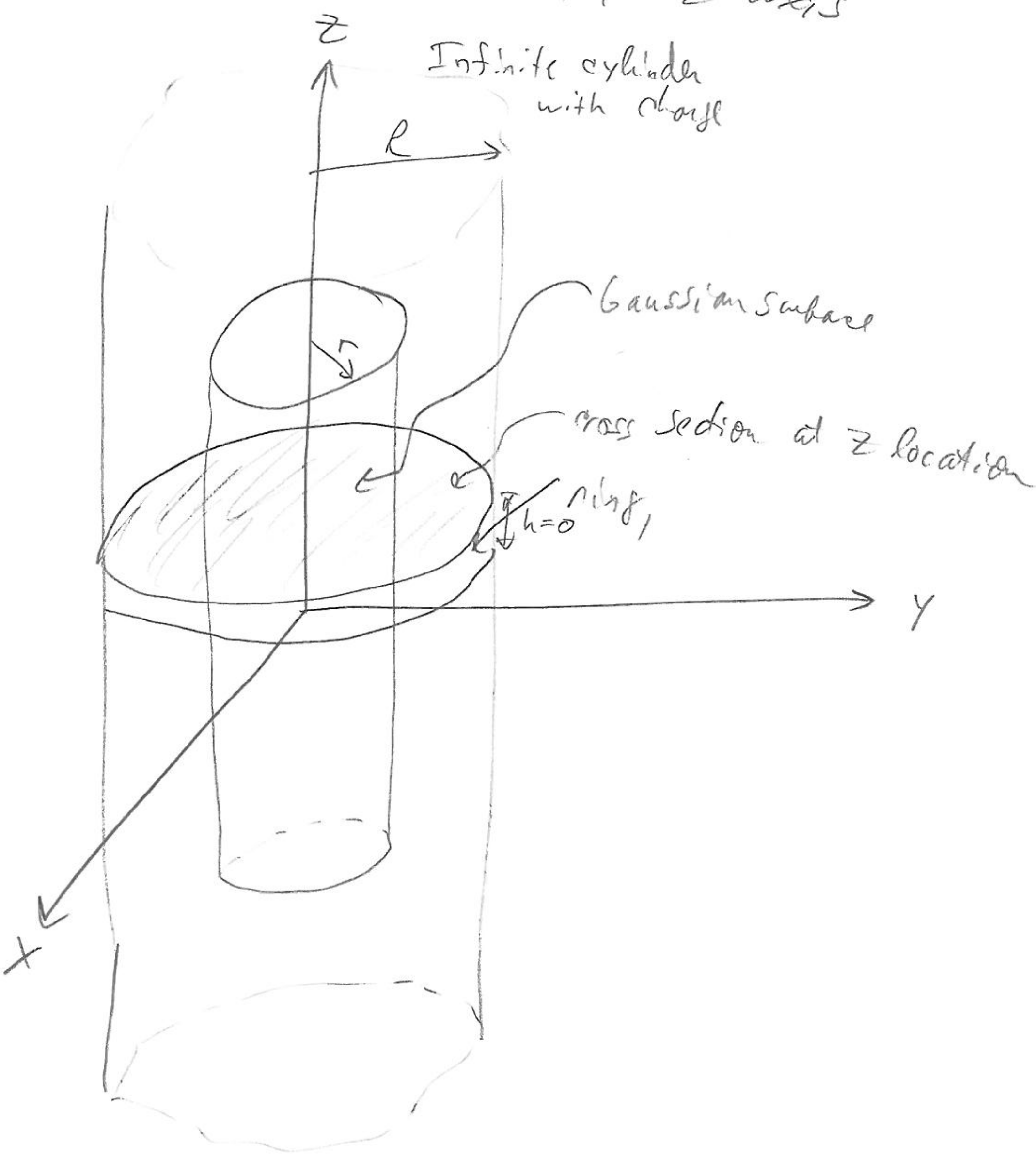
$$d\Psi = \bar{D} \cdot d\bar{S} = (12\hat{x} - 14\hat{y} + 8\hat{z}) \cdot (1E-6)(0.6\hat{x} + 0.8\hat{y})$$

$$= (1E-6)\left[12(0.6) - 14(0.8) + 8(0)\right]$$

$$= 1E-6\left[-4.0\right] = \boxed{-4\,\mu C = d\Psi}$$

3.13) Use Gauss' Law to show that $\bar{D}$ and E are zero at all points in the plane of a uniformly charged circular ring that are inside the ring.

- in order to use Gauss' law, a Gausian surface must be created inside of the circular ring
  - Gaussian surfaces are 3D and completely enclosed
  - For this, we will have to make an infinite cylinder with a charge, put a closed Gaussian surface inside, then take a cross section accross the z axis



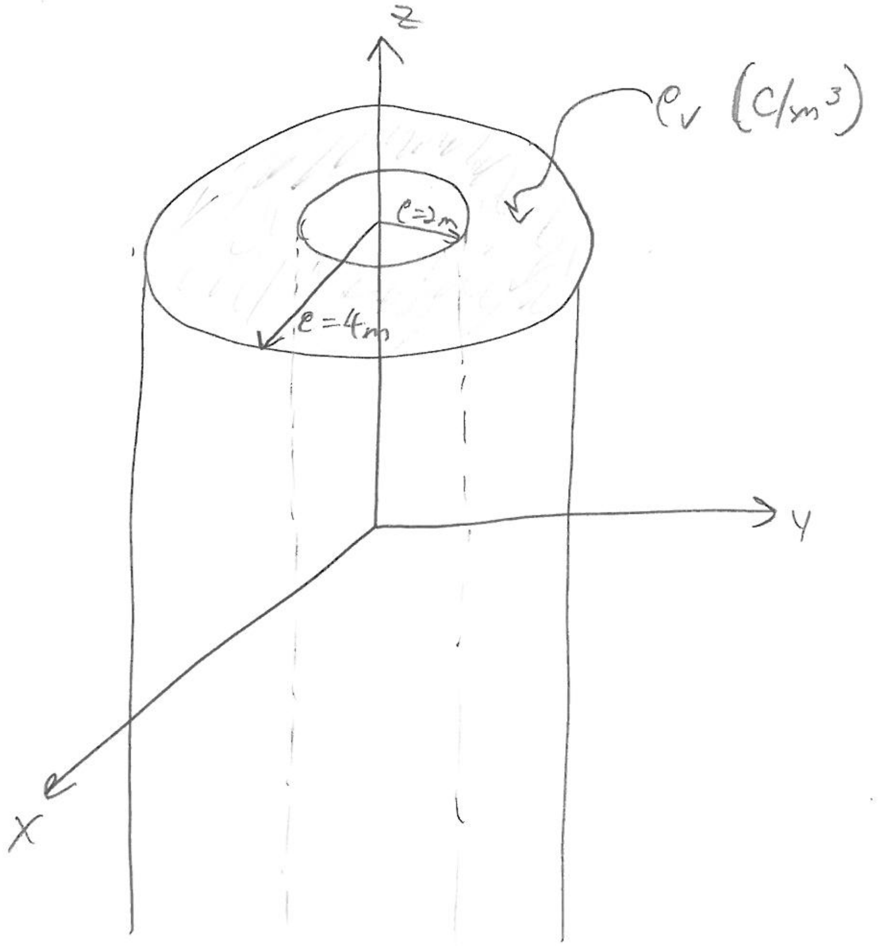
- for the Gaussian surface,

$Q_{enc} = \oint \bar{D} \cdot d\bar{S} = 0$, because there is no enclosed charge,

- hence $\bar{D} = 0$ must be occuring because $d\bar{S}$ is non-zero
- this only occurs for $r < R$,
  if $r > R$ then the Gaussian surface would be outside of the charged ring, enclosing it,
- and, since Ψ (net electric flux) is in the radial ($\hat{\rho}$) direction, if a cross section were taken (causing a ring) $\bar{D}$ would still be 0
- since $\bar{D} = 0$, then $\bar{D} = \varepsilon E \Rightarrow E = 0$ also

shown

3.15) The volume in cylindrical coords between $\rho = 2m$ and $\rho = 4m$ contains a uniform charge density ρ_v (C/m³). Use Gauss' law to find $\bar{D}$ in all regions.



- break it down into 3 regions: $0 < \rho < 2$, $2 < \rho < 4$, $4 < \rho$

- for $0 < \rho < 2$

$Q_{enc} = D \oint dS$ $2\pi r L$ = Area of a cylinder

$Q_{enc} = D(2\pi \rho L)$ eq 1

Since the charge enclosed inside of $\rho < 2$ is 0,

$Q_{enc} = 0$ for $\rho < 2$ $\Rightarrow 0 = D(2\pi\rho L)$

↳ Since there is no charge density here

$\Rightarrow \boxed{D = 0}$ for $\rho < 2$

– for $2 < \rho < 4$

$$Q = \int_V \rho_v dv = \int_{z=0}^{L}\int_{\phi=0}^{2\pi}\int_{\rho=2}^{4} \rho_v \, \rho \, d\rho \, d\phi \, dz$$

$$= \rho_v \int_0^L \int_0^{2\pi} \left[\frac{\rho^2}{2}\right]_2^4 d\phi \, dz = \cancel{\rho_v \int_0^L \int_0^{2\pi} \left[\frac{4^2}{2} - \frac{2^2}{2}\right] d\phi \, dz}$$

$$\cancel{= 6\rho_v \int_0^L [2\pi - 0] \, dz = 2\pi(6)\rho_v L = 12\pi \rho_v L}$$

up to but not including $\rho = 4$, that is why we select a var here

$$\Rightarrow \rho_v \int_0^L \int_0^{2\pi} \left[\frac{\rho^2}{2} - \frac{4}{2}\right] d\phi \, dz = \frac{\rho_v}{2} \int_0^L \int_0^{2\pi} [\rho^2 - 4] \, d\phi \, dz$$

$$= 2\pi \left(\frac{\rho_v}{2}\right) \int_0^L [\rho^2 - 4] \, dz = \pi \rho_v L (\rho^2 - 4) = Q$$

– apply eq 1

$$Q_{enc} = \pi \rho_v L (\rho^2 - 4) = D(2\pi \rho L)$$

$$\Rightarrow D = \frac{\pi \rho_v L (\rho^2 - 4)}{2\pi \rho L}$$

$$D = \frac{\rho_v}{2\rho}(\rho^2 - 4)$$

– and, since the ~~[illegible]~~ normal comp of the cylinder is $\hat{\rho}$, then

$$\bar{D} = \left(\frac{\rho_v}{2\rho}\right)(\rho^2 - 4)\hat{\rho} \quad \text{for } 2 < \rho < 4$$

– for $\rho > 4$

upto and including 4, since we are integrating over all charge

$$Q = \int_V \rho_v dv = \int_{z=0}^{L}\int_{\phi=0}^{2\pi}\int_{\rho=2}^{4} \rho_v \rho \, d\rho \, d\phi \, dz$$

$$= \rho_v \int_0^L \int_0^{2\pi} \left[\frac{\rho^2}{2}\right]_2^4 d\phi \, dz = \rho_v \int_0^L \int_0^{2\pi} \left(\overset{8}{\cancel{\tfrac{4^2}{2}}} - \overset{2}{\cancel{\tfrac{2^2}{2}}}\right) d\phi \, dz$$

$$= 6\rho_v \int_0^L 2\pi \, dz = 12\pi\rho_v L = Q$$

– apply eq 1

$$Q = D(2\pi\rho L) = 12\pi\rho_v L$$

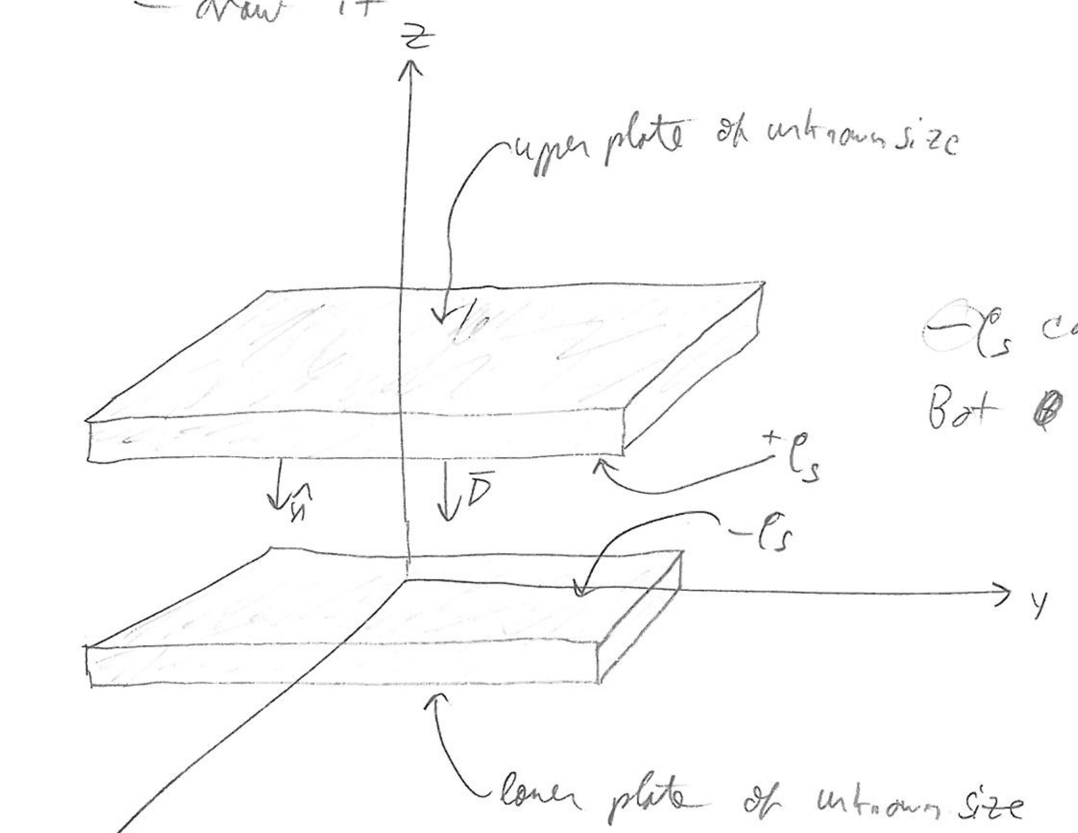
$$\Rightarrow D = \frac{12\pi\rho_v \cancel{L}}{2\pi\rho \cancel{L}} = \frac{6\rho_v}{\rho}$$

– and since the normal comp. of the cylindrical charged volume is $\hat{\rho}$,

$$\boxed{\bar{D} = \left(\frac{6\rho_v}{\rho}\right)\hat{\rho} \quad \text{for } \rho > 4}$$

3.17) A parallel-plate capacitor has a surface charge on the lower side of the upper plate of $+\rho_s$ (C/m²). The upper surface of the lower plate contains $-\rho_s$ (C/m²). Neglecting fringing and use Gauss' law to find $\bar{D}$ and $\bar{E}$ in the region between the plates.

- draw it



$-\rho_s$ cancels $+\rho_s$ at Bot plate

- Since we are looking at the region only between the plates ~~we can~~
  - we can ignore all fringing
  - only have one $\bar{D}$ to deal with

$$Q_{enc} = \cancel{\oint_{top} \bar{D} \cdot d\bar{s}} + \oint_{Bot} \bar{D} \cdot d\bar{s} + \cancel{\oint_{sides} \bar{D} \cdot d\bar{s}}$$

charge density is only on Bot plate

0 no fringing

$$Q_{enc} = \oint_{Bot} \bar{D} \cdot d\bar{s}$$

$$Q = \int_S \rho_s \, ds = \rho_s A$$

area of the plate, result of the integration

$$Q = \rho_s A,$$

$$Q_{enc} = Q = \oint \bar{D} \cdot d\bar{S} = \oint D \, ds = DA$$

$$Q = DA$$

$$\rho_s A = DA \Rightarrow D = \rho_s$$

- D is pointed in the $\hat{n}$ direction

$$\boxed{\bar{D} = \rho_s \hat{n}, \quad \bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{\rho_s}{\varepsilon_0}\hat{n}}$$

done

Chapter 4: Divergance and the Divergance Theorem

4.1) Develop the expression for divergance in cylindrical coordinates:

4.3) Show that the $\bar{D}$ field due to a pt. charge has a divergance of $\emptyset$:

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \Rightarrow \bar{D} = \varepsilon E = \frac{Q}{4\pi r^2}\hat{r} = \bar{D}$$

(in spherical coords)

$$\nabla\cdot\bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Q\cancel{r^2}}{4\pi\cancel{r^2}}\right) = \frac{1}{r^2}(0) = \boxed{0}$$

4.5) Given $\bar{A} = x^2\hat{x} + yz\hat{y} + xy\hat{z}$, find $\nabla\cdot\bar{A}$

$$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial(x^2)}{\partial x} + \frac{\partial(yz)}{\partial y} + \frac{\partial(xy)}{\partial z} = 2x + z + 0$$

4.7) Given $\bar{A} = \rho\sin\phi\,\hat{\rho} + 2\rho\cos\phi\,\hat{\phi} + 2z^2\hat{z}$, find $\nabla\cdot\bar{A}$

$$\nabla\cdot\bar{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho^2\sin\phi\right] + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left[2\rho\cos\phi\right] + \frac{\partial}{\partial z}\left[2z^2\right]$$

$$= \frac{2\cancel{\rho}\sin\phi}{\cancel{\rho}} - 2\sin\phi + 4z = \boxed{4z}$$

4.9) Given $\bar{A} = \frac{5}{r^2}\hat{r} + \frac{10}{\sin\theta}\hat{\theta} - r^2\phi\sin\theta\hat{\phi}$, find $\nabla\cdot\bar{A}$

$$\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\cancel{r^2}\frac{5}{\cancel{r^2}}\right] = \frac{1}{r^2}(0) = 0$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\cancel{\sin\theta}\frac{10}{\cancel{\sin\theta}}\right) = 0$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[-r^2\phi\sin\theta\right] = \frac{1}{r\sin\theta}\left[-r^2\sin\theta\right] = -r$$

$$\boxed{\nabla\cdot\bar{A} = -r}$$

4.11) Given that $\bar{D} = \rho_0 z\hat{z}$ in the region $-1 \le z \le 1$ in cartesian coordinates and $\bar{D} = \frac{\rho_0 z}{|z|}\hat{z}$ elsewhere, find the charge density:

- for $-1 \le z \le 1$: $\nabla\cdot\bar{D} = \rho = \frac{\partial}{\partial z}(\rho_0 z) = \boxed{\rho_0}$

- for all other z: $\nabla\cdot\bar{D} = \rho = \frac{\partial}{\partial z}\left(\frac{\rho_0 z}{|z|}\right) = \frac{\partial}{\partial z}(\pm\rho_0) = \boxed{0}$

4.13) Given that

$$\bar{D} = \frac{Q}{\pi r^2}(1-\cos(3r))\,\hat{r}$$

in spherical coordinates, find the charge density:

$$\rho = \nabla\cdot\bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{Q}{\pi r^2}(1-\cos 3r)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{Q}{\pi}(1-\cos 3r)\right] = \frac{Q}{\pi r^2}\frac{\partial}{\partial r}(1-\cos 3r)$$

$$= \frac{Q}{\pi r^2}\left[0 + 3\sin(3r)\right] = \frac{Q\,3\sin(3r)}{\pi r^2} = \rho$$

4.15) In the region $r \leq 2$, $\bar{D} = \frac{5r^2}{4}\hat{r}$, and for $r > 2$, $\bar{D} = \frac{20}{r^2}\hat{r}$, in spherical coordinates. Find the charge density:

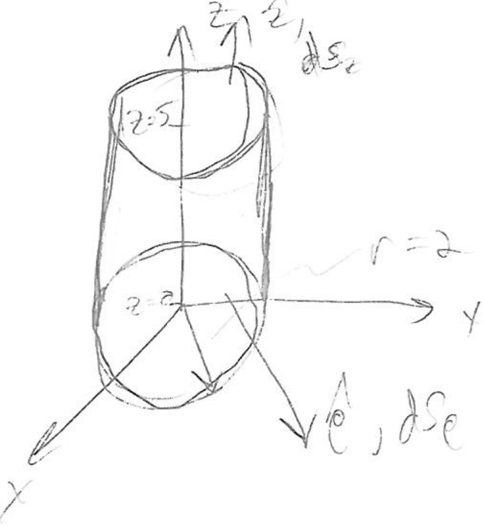
- for $r \leq 2$, $\rho = \nabla\cdot\bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{5r^2}{4}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{5r^4}{4}\right)$

$$= \frac{1}{r^2}\left[\frac{20r^3}{4}\right] = \frac{1}{r^2}\left[5r^3\right] = \boxed{5r}$$

- for $r > 2$,

$$\rho = \nabla\cdot\bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{20}{r^2}r^2\right] = \boxed{0}$$

4.17) Given that $\bar{A} = 30e^{-\rho}\hat{\rho} - 2z\hat{z}$ in cylindrical coordinates, evaluate both sides of the divergance theorem for the volume enclosed by $\rho=2$, $z=0$, and $z=5$



- the divergance theorem:

$$\oint_S \bar{A}\cdot d\bar{s} = \int_V (\nabla\cdot\bar{A})\,dv$$

→ evaluate the left side first

$$\oint_S \bar{A}\cdot d\bar{s}$$

where $d\bar{s} = dS_\rho\hat{\rho} + dS_z\hat{z} = \left(\rho\, d\phi\, dz\,\hat{\rho} + \rho\, d\rho\, d\phi\,\hat{z}\right)$

$$\oint_S [30e^{-\rho}\hat{\rho} - 2z\hat{z}]\cdot[\rho\, d\phi\, dz\,\hat{\rho} + \rho\, d\rho\, d\phi\,\hat{z}]$$

$$= \int_{z=0}^{5}\int_{\phi=0}^{2\pi} 30e^{-\rho}\rho\, d\phi\, dz - \int_{\phi=0}^{2\pi}\int_{\rho=0}^{2} 2z\rho\, d\rho\, d\phi$$

subst $\rho=2$ since it is 2 and not changing in this integral

subst $z=5$,

4

$$= 2(30)e^{-2}\int_{z=0}^{5}\int_{\phi=0}^{2\pi} d\phi\, dz - 1(10)\int_{\phi=0}^{2\pi}\int_{\rho=0}^{2} \rho\, d\rho\, d\phi$$

$$= 60e^{-2}(2\pi)5 - 10\int_{\phi=0}^{2\pi}\left[\frac{\rho^2}{2}\right]_0^2 d\phi$$

$$= 255.1 - 10\left[\frac{4}{2} - 0\right]2\pi = \boxed{129.44} \quad (1)$$

evaluate the right side next

$$\int_V (\nabla \cdot \bar{A})\, dV$$

$$\nabla \cdot \bar{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho 30e^{-\rho}\right] + \frac{\partial}{\partial z}\left[-2z\right] = \frac{1}{\rho}\left[30e^{-\rho} - \rho 30e^{-\rho}\right] - 2$$

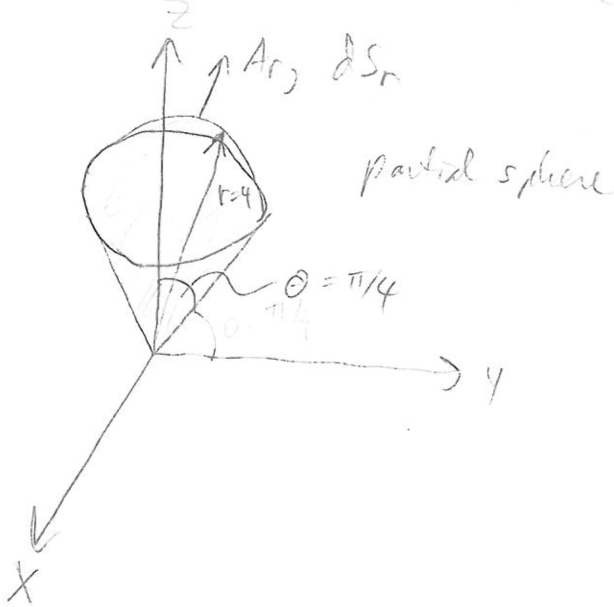
$$= \frac{30e^{-\rho}}{\rho} - 30e^{-\rho} - 2$$

$$\iint (\nabla \cdot \bar{A})\, dv = \int_{z=0}^{5}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{2}\left[\frac{30e^{-\rho}}{\rho} - 30e^{-\rho} - 2\right]\rho\, d\rho\, d\phi\, dz$$

$$= \int_0^5\int_0^{2\pi}\int_0^2\left[30e^{-\rho} - \rho 30e^{-\rho} - 2\rho\right] d\rho\, d\phi\, dz$$

$$= \int_0^5\int_0^{2\pi} 4.12\, d\phi\, dz = \boxed{129.43} \quad \text{same as } (1)$$

4.19) Given that $\bar{D} = \frac{5r^2}{4}\hat{r}$ (C/m²) in spherical coords, evaluate both sides of the divergence theorem for the volume enclosed by $r = 4$m, $\theta = \pi/4$



- the divergence theorem

$$\oint_S \bar{A}\cdot d\bar{S} = \int_V (\nabla\cdot\bar{A})\,dV$$

- evaluate the left side first

$$\oint_S \bar{D}\cdot d\bar{S} = \oint_S \left(\frac{5r^2}{4}\hat{r}\right)\cdot\left(r^2\sin\theta\, d\theta\, d\phi\,\hat{r}\right)$$

$$d\bar{S} = dS_r = r^2\sin\theta\, d\theta\, d\phi\,\hat{r}$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{5r^2}{4} r^2 \sin\theta\, d\theta\, d\phi = \frac{5\,4^2\cdot 4^2}{4}\int_0^{2\pi}\int_0^{\pi/4}\sin\theta\, d\theta\, d\phi$$

$r = 4$

$$= 320\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi/4} d\phi = 320\left[-0.707+1\right]2\pi = \boxed{589.1\ C}$$

- evaluate the right side

$$\int_V (\nabla \cdot \bar{A})\, dv$$

$$\nabla \cdot \bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{5r^2}{4}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{5r^4}{4}\right)$$

$$= \frac{1}{r^2}(5r^3) = 5r$$

$$\int_V (\nabla \cdot \bar{D})\, dv = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{4} 5r\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

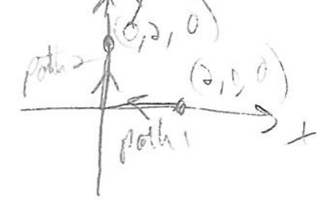
$$= 5\int_0^{2\pi}\int_0^{\pi/4}\int_0^{4} r^3 \sin\theta\, dr\, d\theta\, d\phi = 5\int_0^{2\pi}\int_0^{\pi/4}\left[\frac{r^4}{4}\right]_{r=0}^{4} \sin\theta\, d\theta\, d\phi$$

$$= 5\frac{(4^4)}{4}\int_0^{2\pi}\int_0^{\pi/4} \sin\theta\, d\theta\, d\phi = 320\left[-\cos\theta\right]_0^{\pi/4} 2\pi$$

$$= 320(2\pi)\left[-0.707+1\right] = \boxed{589.11\ C}$$ Same as above

Ch 5: Electrostatic Field: Work, Energy, and Potential

5.1) Given the electric field $E = 2x\hat{x} - 4y\hat{y}$ (V/m) find the work done in moving a point charge $+2\,C$



$\boxed{\text{for path 1}}$

a) from $(2,0,0)$ m to $(0,0,0)$

$dl_1 = \hat{x}$ because it only moves along the x axis (from 2 to 0)

$dw_1 = -QE \cdot d\bar{l}_1 = -2\left[(2x\hat{x} - 4y\hat{y}) \cdot (+1\hat{x})\right]$

$= -2(2x) = \boxed{-4x}$

$\boxed{\text{for path 2}}$

from $(0,0,0)$ to $(0,2,0)$

$dl_2 = +\hat{y}$ because it only moves along the y axis (from 0 to 2)

$dw_2 = -QE \cdot dl_2 = -2\left[(2x\hat{x} - 4y\hat{y}) \cdot \hat{y}\right]$

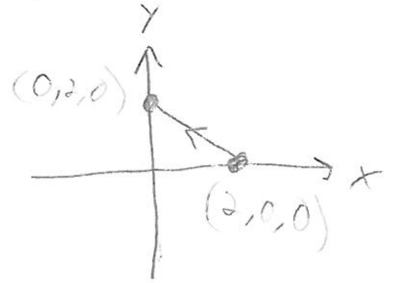
$= -2[-4y] = \boxed{8y}$

Combine the 2 results:

$$W = -4\int_2^0 x\,dx + 8\int_0^2 y\,dy = -4\left[\frac{x^2}{2}\right]_2^0 + 8\left[\frac{y^2}{2}\right]_0^2$$

$$= -4[0-2] + 8[2-0] = 8 + 16 = \boxed{24\ J}$$

b) from $(2,0,0)$ to $(0,2,0)$ along the straight line path.



- need to use parametric eq's for a direct line integral

$x = 2 - 2t$ for $t = 0$ to 1, $\quad y = 2t$ for $t = 0$ to 1, $\quad z = 0$

$\Rightarrow dx = -2\,dt, \quad dy = 2\,dt$

subst, subst

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} = -2\,dt\,\hat{x} + 2\,dt\,\hat{y}$$

subst, subst

$$\vec{E} = 2x\,\hat{x} - 4y\,\hat{y} = 2(2-2t)\,\hat{x} - 4(2t)\,\hat{y}$$

$$dw = -Q\vec{E}\cdot d\vec{l} = -2\left[\left(2(2-2t)\hat{x} - 4(2t)\hat{y}\right)\cdot\left[-2\,dt\,\hat{x} + 2\,dt\,\hat{y}\right]\right]$$

$$= -2\left[-4(2-2t)\,dt - 8(2t)\,dt\right]$$

$$dw = 8(2-2t)\,dt + 16(2t)\,dt$$

$$= (16 - 16t)\,dt + 32t\,dt$$

$$= (16 - 16t + 32t)\,dt = (16 + 16t)\,dt$$

$$dw = 16(1+t)\,dt$$

$$w = \int dw = 16\int_{t=0}^{1}(1+t)\,dt = 16\left[t + \frac{t^2}{2}\right]_0^1$$

$$= 16\left[1 + \frac{1}{2}\right] = \boxed{24\,J}$$

5.3) Given the field $E = \frac{k}{r}\hat{r}$ in cylindrical coords, show that the work needed to move a pt. charge Q from any radial distance r to a pt. at twice that radial distance is independent of r!

- Since the field is only in terms of r

$$d\bar{l} = dr\,\hat{r}$$

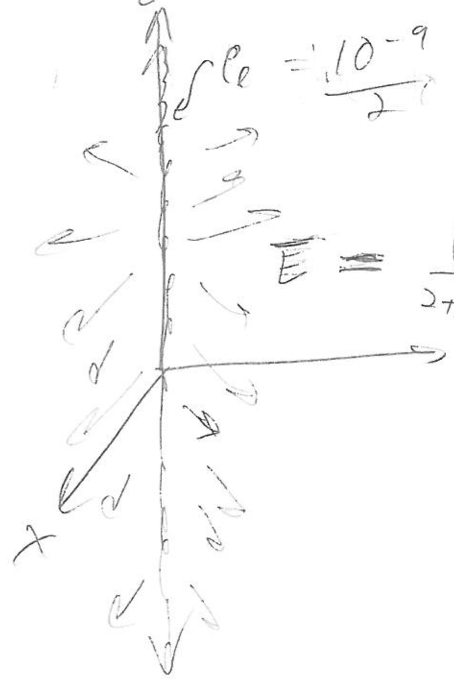
$$\Rightarrow dW = -Q\left[\frac{k}{r}\hat{r}\cdot dr\,\hat{r}\right] = -Q\frac{k}{r}dr$$

$$W = \int dW = -Q\int_{r_1}^{2r_1}\frac{k}{r}dr = -Qk\Big[\ln r\Big]_{r_1}^{2r_1}$$

$$W = -Qk\left[\ln(2r_1) - \ln(r_1)\right]$$

$$= -Qk\ln\left(\frac{2r_1}{r_1}\right) = \boxed{-Qk\ln 2 = W}$$

5.5) for a line charge $\rho_l = \frac{10^{-9}}{2}$ C/m on the z axis, find V_{BC}, where $r_B = 4$m and $r_C = 10$m



$\overline{E} = \frac{\rho_l}{2\pi\varepsilon_0|r|}\hat{r}$ ← E field due to a line charge

$d\overline{l} = dr\,\hat{r}$

$$V_{BC} = -\int_C^B \overline{E}\cdot d\overline{l} = -\int_{r=10}^{r=4} \frac{\rho_l}{2\pi\varepsilon_0|r|}\hat{r}\cdot dr\,\hat{r}$$

$$= \frac{-\rho_l}{2\pi\varepsilon_0}\int_{10}^{4} \frac{1}{r}dr = \frac{-\rho_l}{2\pi\varepsilon_0}\left[\ln r\right]_{10}^{4} = -8.99[\ln 4 - \ln 10]$$

$\rho_l = 10^{-9}/2$

$\varepsilon_0 = 8.854E{-12}$

$$= -8.99 \ln\left(\frac{4}{10}\right)$$

$$= \boxed{+8.24v}$$

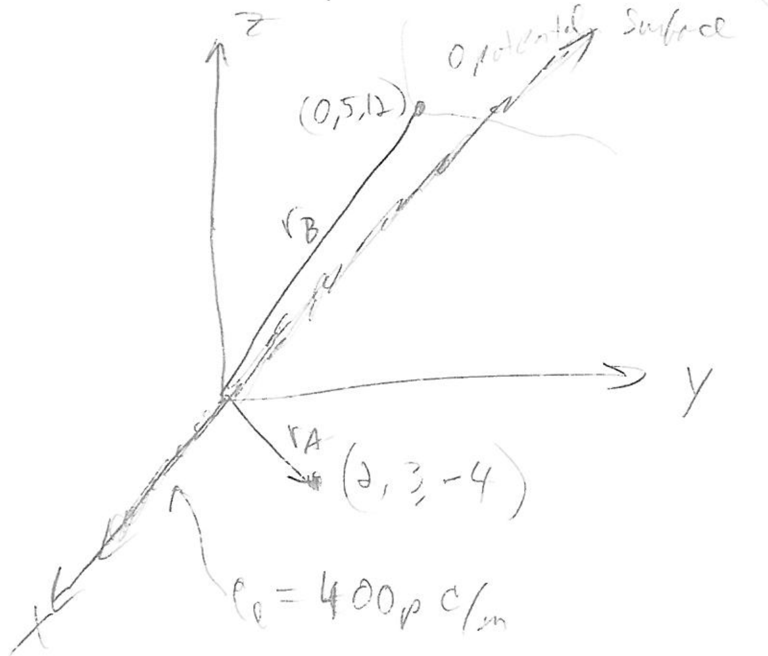
- next, find V_{AC}

$$V_{AC} = -8.99 \ln\left(\frac{2}{10}\right) = \boxed{14.47v}$$

A→ $A_r = 2$

$C_r = 10$

5.7) A line charge $\rho_\ell = 400\ pC/m$ lies along the x axis and the surface of zero potential passes through the pt. $(0,5,12)$ m in cartesian coords. Find the potential at $(2,3,-4)$ m



- Since the line charge is infinite along the x axis, the x coords can be ignored

thus $r_B = \sqrt{5^2+12^2} = 13$, $r_A = \sqrt{3^2+(-4)^2} = 5$

$E = \dfrac{\rho_\ell}{2\pi\varepsilon_0 r}\hat{e}$, $d\bar{\ell} = dr\,\hat{e}$

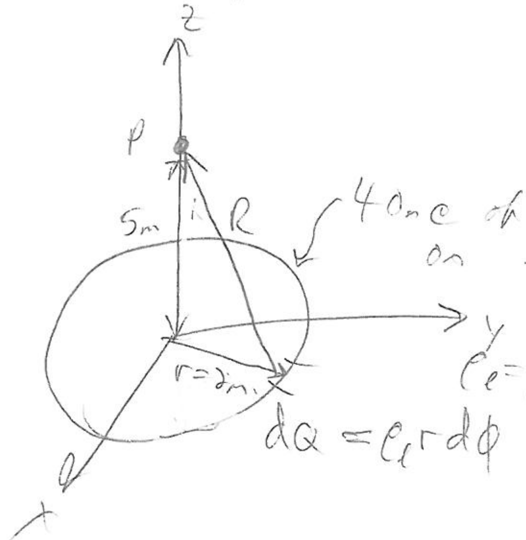
$\Rightarrow$ find the potential from pt A to pt B

$$V_{AB} = -\int_B^A \bar{E}\cdot d\bar{\ell} = \underbrace{-\frac{\overset{400p}{\rho_\ell}}{2\pi\underset{8.854E-12}{\varepsilon_0}}}_{-7.19}\int_{13}^{5}\frac{1}{r}\,dr = -7.19\Big[\ln r\Big]_{13}^{5}$$

$$= -7.19\left[\ln 5 - \ln 13\right]$$

$$= -7.19\ln(5/13) = \boxed{6.87\ V}$$ 5

5.9) 40nC of charge is uniformly distributed around a circular ring of r = 2m, Find the potential at a pt. on the axis 5m from the plane of the ring.



⇒ charge density of the ring is

$\rho_\ell = \frac{Q}{2\pi r} = \frac{40n}{2\pi(2)} = \boxed{3.18E-9\ C/m = \rho_\ell}$

$dQ = \rho_\ell r\, d\phi$

$R = \sqrt{5^2+2^2} = 5.385 = R$

$d\ell = r\, d\phi$

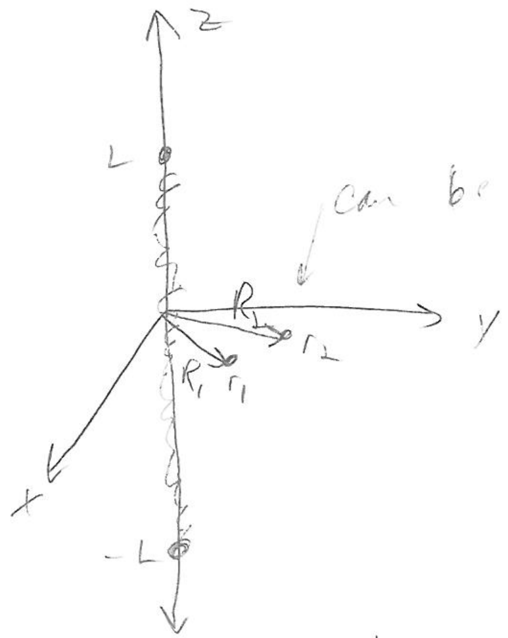
$$V = \int \frac{\rho_\ell\, d\ell}{4\pi\varepsilon_0 R} = \int_{\phi=0}^{2\pi} \frac{\rho_\ell r\, d\phi}{4\pi\varepsilon_0 R} = \int_0^{2\pi} \frac{(3.18E-9)(2)\, d\phi}{4\pi(8.854E-12)(5.385)}$$

$$V = 10.61 \int_0^{2\pi} d\phi = \boxed{66.69\ V = V}$$

– compare w/the result where all the charge is at the origin in the form of a pt. charge:

$$V = \frac{Q}{4\pi\varepsilon_0 r} = \frac{40n}{4\pi(8.854E-12)5} = \boxed{71.9\ V}$$

5.11) Charge is distributed uniformly along a straight line of finite length 2L. Show that for 2 external pts near the midpoint, such that r_1 and r_2 are small compared to the length, the potential V_{12} is the same as for an infinite line charge



so $R_1 = \sqrt{z^2 + r_1^2}$

$R_2 = \sqrt{z^2 + R_2^2}$

$$V_1 = \int_l \frac{\rho_l \, dl}{4\pi\varepsilon_0 R_1} = 2\int_0^L \frac{\rho_l \, dz}{4\pi\varepsilon_0 R_1} = \frac{2\rho_l}{4\pi\varepsilon_0}\int_0^L \frac{1}{\sqrt{z^2+R_1^2}}\,dz$$

$$= \frac{2\rho_l}{4\pi\varepsilon_0}\Big[\ln\left(\sqrt{z^2+R_1^2} + z\right)\Big]_0^L$$

$$V_1 = \frac{2\rho_l}{4\pi\varepsilon_0}\left[\ln\left(L + \sqrt{L^2+R_1^2}\right) - \ln R_1\right]$$

$$\Rightarrow V_2 = \frac{2\rho_l}{4\pi\varepsilon_0}\left[\ln\left(L + \sqrt{L^2+R_2^2}\right) - \ln R_2\right]$$

$\longrightarrow$

– Now, if L is much larger than R_1 and R_2

$L \gg R_1$, $L \gg R_2$

$$\Rightarrow V_1 = \frac{2 \rho_l}{4\pi\varepsilon_0}\left[\ln[2L] - \ln R_1\right]$$

$$\Rightarrow V_2 = \frac{2 \rho_l}{4\pi\varepsilon_0}\left[\ln[2L] - \ln R_2\right]$$

thus $V_{12} = V_1 - V_2 = \frac{2\rho_l}{4\pi\varepsilon_0}\left[\cancel{\ln[2L]} - \ln R_1 - \cancel{\ln[2L]} + \ln R_2\right]$

$$V_{12} = \frac{2\rho_l}{4\pi\varepsilon_0}\ln\left(\frac{R_2}{R_1}\right)$$

which is identical for the expression for a infinite line (shown in 5.7)

5.13) Given the potential function $V = 2x + 4y$ (V) in free space, find the stored energy in a $1 m^3$ volume centered at the origin:

$V = 2x + 4y$, $\nabla V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$

$E = -\nabla V = -\left[\frac{\partial (2x+4y)}{\partial x}\hat{x} + \frac{\partial (2x+4y)}{\partial y}\hat{y} + \cancel{\frac{\partial (2x+4y)}{\partial z}\hat{z}}\right]$

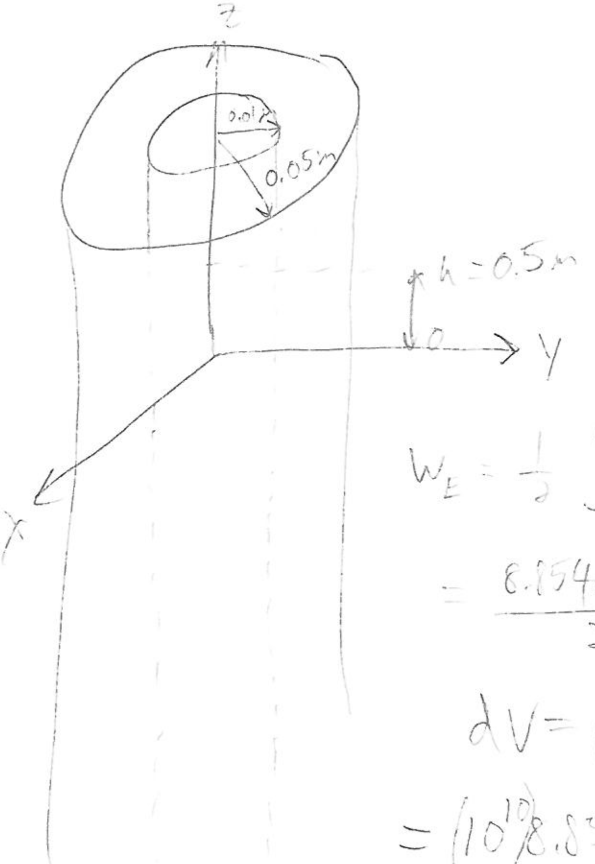
$E = -\left[2\hat{x} + 4\hat{y}\right] = -2\hat{x} - 4\hat{y} = E$

$W_E = \frac{1}{2}\int \varepsilon E^2 dV = \frac{(8.854E-12)(4.47)}{2}\int_0^1\int_0^1\int_0^1 dx\,dy\,dz$

$E = \sqrt{(-2)^2 + (4)^2} = 4.47$

$\boxed{W_E = 19.79 \, pC}$

5.15) The electric field between 2 concentric cylindrical conductors at $r = 0.01m$ and $r = 0.05m$ is given by $E = \frac{10^5}{\rho}\hat{\rho}$, fringing neglected. Find the energy stored in a 0.5 m length:



$E = \frac{10^5}{\rho}\hat{\rho}$

$\Rightarrow E = \frac{10^5}{\rho}$

$W_E = \frac{1}{2}\int \varepsilon E^2 dv$

$= \frac{8.854E-12}{2}\int \left(\frac{10^5}{\rho}\right)^2 dV$

$dV = \rho\, d\rho\, d\phi\, dz$ in cylindrical coords

$= \frac{(10^{10})8.854E-12}{2}\iiint \frac{1}{\rho^2}\rho\, d\rho\, d\phi\, dz$

$= 44.27E-3 \int_{z=0}^{0.5}\int_{\phi=0}^{2\pi}\int_{\rho=0.01}^{0.05} \frac{1}{\rho}\, d\rho\, d\phi\, dz = 44.27E-3 \int_0^{0.5}\int_0^{2\pi} \left[\ln \rho\right]_{0.01}^{0.05} d\phi\, dz$

$= 44.27E-3 \int_0^{0.5}\int_0^{2\pi} \ln\left(\frac{0.05}{0.01}\right) d\phi\, dz = (44.27E-3)\ln\left(\frac{0.05}{0.01}\right) 2\pi(0.5)$

$= \boxed{0.224\ J}$

5.17) What energy is stored in the system of 2 pt charges $Q_1 = 3n\,C$ and $Q_2 = -3n\,C$, separated by a distance of $d = 0.2\,m$?

$$W_E = \frac{1}{2} \sum_{m=1}^{n} Q_m V_m$$

$$= \frac{1}{2}\left[3n \left(\frac{-3n}{4\pi\varepsilon_0 (0.2)} \right) - 3n \left(\frac{3n}{4\pi\varepsilon_0 (0.2)} \right) \right]$$

$$= \frac{-3n\,3n}{4\pi\varepsilon_0 (0.2)} = -404.45\,nJ$$

Ch6: Current, Current Density, and Conductors

6.1) An AWG #12 copper conductor has an 80.8 mil diameter. A 50 ft length carries a current of 20 A. Find the electric field intensity E, drift velocity U, the voltage drop, and the resistance for the 50 ft length:

- For copper $\sigma = 5.8E+7 \rightarrow \mu = 0.0032$
- $1\,in = 2.54E-2\,m$

$$J = \sigma E \Rightarrow E = \frac{J}{\sigma} = \frac{6.042E6}{5.8E+7} = \boxed{0.104\ V/m}$$

$$J = \frac{I}{A} = \frac{20}{3.31E-6} = 6.042E6$$

$$A = \pi r^2 = \pi\left(\frac{0.0808}{2} \cdot \frac{2.54E-2}{1\,in}\right)^2 = 3.31E-6\ m^2$$

$$U = \mu E = 0.0032[0.104] = \boxed{332.8\mu = U}$$

$$E = \frac{V}{l} \Rightarrow V = El = [0.104][15.24] = \boxed{1.58\ V}$$

$$l = (50)(12)(2.54E-2) = 15.24\,m$$

$V = IR$

$$\Rightarrow R = \frac{V}{I} = \frac{1.58}{20} = \boxed{79\,m\,R}$$

6.5) What is the density of free e^-'s in a metal for a mobility of $\mu = 0.0046 \frac{m^2}{V \cdot s}$ and a conductivity $\sigma = 29.1\ MS/m$

- given σ find the charge density ρ

$$\sigma = \rho \mu$$

$$\Rightarrow \rho = \frac{\sigma}{\mu} = \frac{29.1E6}{0.0046} = 6.33E9\ C/m^3$$

$$N_e = \frac{\rho}{e^-} = \frac{6.33E9}{1.6E-19} = 39.56E27\ \text{electrons}/m^3$$

6.7) A conductor of uniform cross section and 150m long has a voltage drop of 1.3V and a current density of 4.65E5 A/m^2. What is the conductivity of the material in the conductor?

$$\sigma = \rho \mu$$

$V = 1.3$

$$E = \frac{V}{l} = \frac{1.3}{150} = 8.67E-3\ V/m$$

$l = 150$

$$J = \sigma E = 4.65E5 = \sigma (8.67E-3)$$

$$\Rightarrow \sigma = 53.63E6\ S/m$$

$J = 4.65E5$

6.9) An AWG #20 Aluminium wire has a resistance of 16.7 ohms per 1000 ft. What conductivity does this imply for aluminum?

#20 wire has d = 0.032"

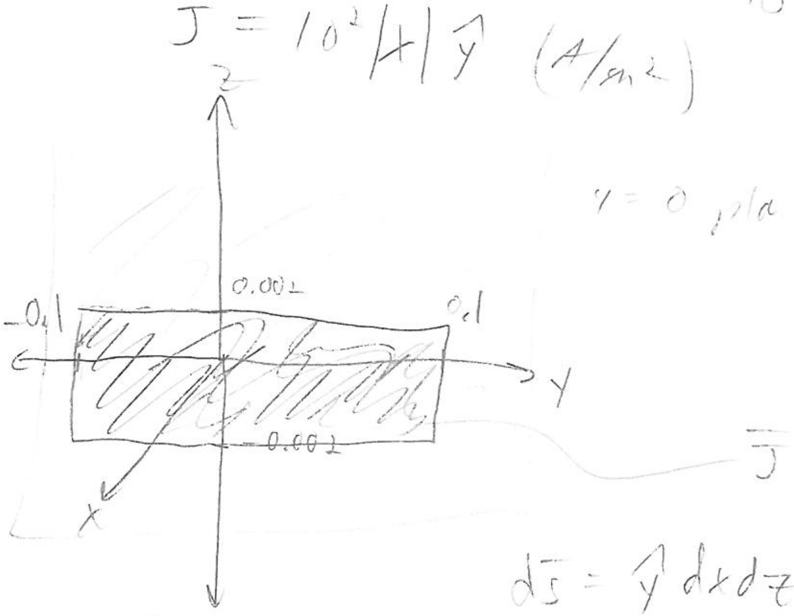
$$R = \frac{l}{\sigma A_n} \Rightarrow 16.7 = \frac{1000 \cdot 12 \cdot 2.54E-2}{\sigma\,(518.9E-9)} \Rightarrow \boxed{\sigma = 35.2E6}$$

$$A = \pi r^2 = \pi\left(\frac{0.032}{2} \cdot 2.54E-2\right)^2 = \cancel{2.075E-6}\ m^2$$

518.9E-9

6.11) Find the current crossing the portion of the y = 0 plane defined by $-0.1 \le x \le 0.1$ m and $-0.002 \le z \le 0.002$ m if

$$\bar{J} = 10^2|x|\,\hat{y}\ (A/m^2)$$



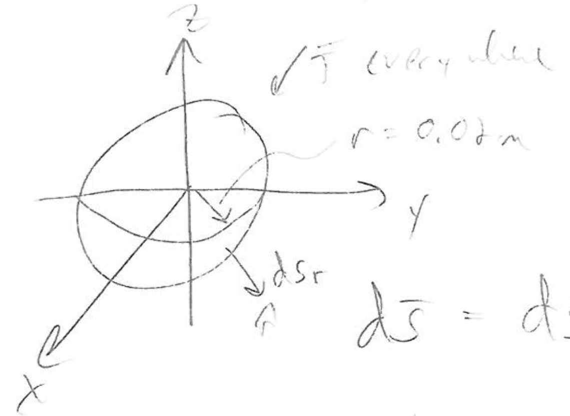
$$d\bar{s} = \hat{y}\,dx\,dz$$

$$I = \int_s \bar{J} \cdot d\bar{s} = \int_s 10^2|x|\,\hat{y} \cdot \hat{y}\,dx\,dz$$

$$= 10^2 \int_{-0.002}^{0.002} \int_{-0.1}^{0.1} |x|\,dx\,dz = 100 \cancel{\int_{-0.002}^{0.002} \left[\frac{x|x|}{2}\right]_{-0.1}^{0.1}} = 10E-3 = \boxed{4\ mA}$$

4

6.13) Given $\bar{J} = 10^3 \sin\theta \, \hat{r}$ A/m², find the current crossing the spherical shell @ $r = 0.02$ m



$d\bar{S} = d\bar{S}_r = \hat{r} r^2 \sin\theta \, d\theta \, d\phi$

$$I = \int_S \bar{J} \cdot d\bar{S} = \int_S 10^3 \sin\theta \, \hat{r} \cdot \hat{r} \, r^2 \sin\theta \, d\theta \, d\phi$$

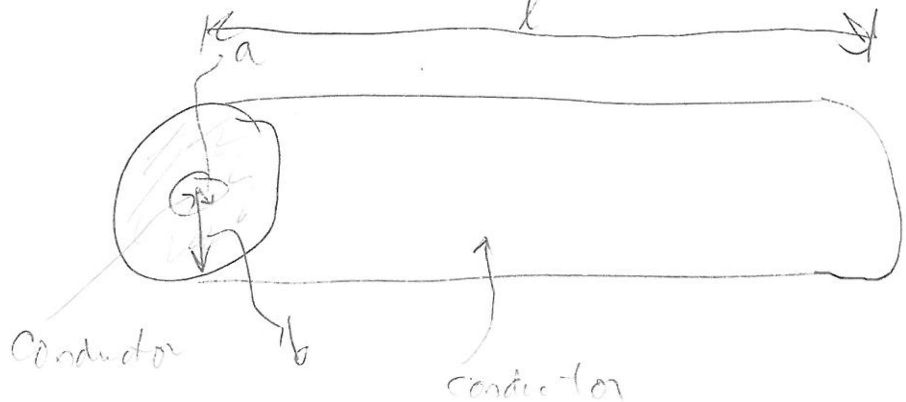
$$= 1000 \int_0^{2\pi} \int_0^{\pi} \sin^2\theta \, r^2 \, d\theta \, d\phi$$

$r = 0.02$ m

$$= 0.4 \int_0^{2\pi} \left[\frac{1}{2} \left(\theta - \sin\theta \cos\theta \right) \right]_0^{\pi} d\phi = 0.4 \int_0^{2\pi} \left[\frac{\pi}{2} \right] d\phi$$

$$= 0.4 \left(\frac{\pi}{2} \right) 2\pi = 0.4\pi^2 = \boxed{3.95 A}$$

6.15) Determine the resistance of the insulation in a length l of coaxial cable:



- assume a total current I from the inner conductor to the outer conductor, then at a radial distance r

$$J = \frac{I}{A} = \frac{I}{2\pi r l}$$

surface area,

$$A = (2\pi r) l$$

$$E = \frac{J}{\sigma} = \frac{I}{2\pi r l \sigma} = E$$

- use the method from the prev. chapter to find V

$$V_{ab} = -\int_B^A \bar{E} \cdot d\bar{l} = -\int_b^a \frac{I}{2\pi r l \sigma} dr = \frac{-I}{2\pi l \sigma} \left[\ln r\right]_b^a$$

$$V_{ab} = \frac{I}{2\pi l \sigma} \ln\left(\frac{b}{a}\right)$$

- and finally, the resistance

$$R = \frac{V}{I} = \frac{\frac{\cancel{I}}{2\pi l \sigma} \ln\left(\frac{b}{a}\right)}{\cancel{I}} = \boxed{\frac{1}{2\pi l \sigma} \ln\left(\frac{b}{a}\right)}$$